FIRST LESSONS

IN

GEOMETRY,

UPON THE

MODEL OF COLBURN'S FIRST LESSONS IN ARITHMETIC.

BY

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PROF. ETC., DART. COLL.

WITH AN INTRODUCTION,

BY

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"It is the nature of genuine science to advance in continual progression. Each step carries it still higher; new relations are descried; and the most distant objects seem gradually to approximate. But, while science thus enlarges its bounds, it likewise tends uniformly to simplicity and concentration. The discoveries of one age are, perhaps, in the next, melted down into the mass of elementary truths. What are deemed at first merely objects of enlightened curiosity become, in due time, subservient to the most important interests. Theory soon descends to guide and assist the operations of practice. To the geometrical speculations of the Greeks, we may distinctly trace whatever progress the moderns have been enabled to achieve in mechanics, navigation, and the various complicated arts of life. A refined analysis has unfolded the harmony of the celestial motions, and conducted the philosopher, through a maze of intricate phenomena, to the great laws appointed for the government of the universe." — PROFESSOR LESLIE.

"The curiosity which speaks in children's busy eyes and hands should be to us the voice of Nature, bidding us make our beginnings early. The infant who cannot speak gazes earnestly and thoughtfully at the most common object, returning to it, and glancing from one part to another, as if to learn their connection. When he can walk, he goes round it, handling it, and studying it with all his senses. When he speaks, his questions are of size, form, and distance. If our answers are careless or unsatisfactory, his quick eyes and mind, not blunted by habit, detect our errors. He loves comparison of objects, and the imaginary multiplication and extension of them; he is pleased with the new and the different, and equally pleased with resemblance and equality to things known before.

"Happy age of natural geometry, — when each look and motion, nay, his very games, lead the boy on to the laws which shape the spheres and hold the planets in their course!" — AUTHOR OF THE THEORY OF TEACHING.

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PREFACE.

THE following work is stated in the title to be upon the model of Colburn's "First Lessons in Arithmetic," because no other method occurred to me of presenting a general view of its plan, which would be at once so brief and so well understood. Without aiming at minute resemblance, and certainly without challenging any comparison in merit, it is an imitation of that admirable work, which has introduced so entire a revolution in the mode of teaching Arithmetic in our country, in the following particulars.

I. It aims to give a *simpler* and *plainer form* to the elements of Geometry, and thus to bring the science, without sacrificing any of that strictness of demonstration which is its peculiar glory, within the province of the common school, and the reach even of the quite young. The relations of place, and form, and magnitude are those of which the youthful, not to say the mature, mind conceives most easily and most distinctly; and it has long seemed to me a subject for regret, that we have had no work in our list of *common-school books* (however excellent treatises may have been prepared for higher institutions, or for special purposes) to take advantage of these juvenile conceptions, to give to them a scientific form, and to make them the foundation of elevated and accurate attainment.

It is obvious that the reasoning employed in Geometry is, in its nature, less abstruse than that which is employed in

Algebra, or even in Arithmetic, if you pass beyond the simpler rules. And it has seemed to me, that no one can consider the admirable discipline which Geometry gives to the mind, its intimate connection with almost all the arts and occupations of men, and its essential concern in those perceptions, comparisons, judgments, calculations, and acts, which, from the very necessity of our nature, constitute the great staple of life, without feeling, that, if the door of the famed school of Plato bore upon the outside the inscription, "Let no one *enter* without a knowledge of Geometry," the door of every common school ought to bear upon the inside the inscription, "Let no one *go forth* without a knowledge of Geometry." Aside from all other advantages of the study, who can compute the vast difference which it makes in the dignity and the pleasure of life, whether we tread its path with an *imperfect conception* or a *distinct view* of the relations and properties of the material objects of grandeur, beauty, and use which are all around us, rising in the distance, skirting our pathway, shining over our heads, and blooming beneath our feet? *Who would be blind, when he may see?*

II. Its method is that of *suggestion*, instead of *dictation*. Its object is to guide and assist the student in discovering for himself the truths of Geometry with their proofs, instead of making his work consist wholly in possessing himself of the precepts and reasonings of another, a work in which, all teachers have observed, the memory has often a larger share than the understanding. That which, more than any thing else, has robbed Geometry of its proper attractiveness, is that the mind has been made too passive in its study. With Americans at least, no study will ever be a favorite, which does not call into exercise the more active and independent powers of the mind. With what interest would Arithmetic or Algebra be studied, if every question were answered as soon as asked, or rather the fact were stated without a question's being even raised, and then this statement

were followed in every instance by the whole operation written out in its minutiæ, and the scholar's work consisted merely in tracing out this operation, of which he already knows the result, and going over it again and again, till he can reproduce it (sometimes with scarce a thought that it has any meaning), when called up for recitation?

III. It abounds in *illustrative questions*, both general and numerical. A truth has become in an especial sense *our own*, when we have learned to apply it. It is now no longer a stranger, but a member of the household. In these questions I have avoided high and complicated numbers, in order that the arithmetical computation might not divert the mind from the geometrical truth to be illustrated.

One important addition to the plan of the work selected as a model will not fail to be observed. It is the full discussion of the elementary ideas of the science, and the definite statement of the results of its investigation in the form of distinct theorems and corollaries. Especial pains have been taken to render the definitions clear, and to conform them to the ideas actually existing in the mind, instead of making them, as they have too often been, disguised presentations of some axiom or theorem. That "a straight line is the shortest distance between two points," that "parallel lines will never meet," and that, "if any two points are taken in a plane, the straight line connecting those points will lie wholly in the plane," are all true enough, and may be readily shown; but, when proposed as definitions, let me ask the thinking man if they do not fail to express the essential idea which the mind has of the object defined, and do not substitute instead of this an elementary proposition in regard to it. I might go still farther, and ask if these definitions have not led in some cases to a species of unconscious sophistry, a reasoning at one time according to the real conception of the mind, and at another according to the arbitrary

definition.* In regard to the theorems, no pains have been spared to make them clear, concise, and comprehensive. In some instances, they have been intentionally so worded as to admit a double application, and to include two propositions, one of which is the converse of the other.

Mathematical signs have been extensively used, from the great relief which they give both to the hand, the eye, and the mind; and from the conviction that every thing which shortens the expression of truth, without sacrificing clearness, assists the mind in apprehending and in retaining it. The plan of the book has led to the adoption of two signs of an interrogative character (§ 13). To assist still farther the eye and the mind, each step in a demonstration is commonly printed in a separate line.

It will be observed that the section-mark is sometimes prefixed rather for convenience of reference than to denote change of subject; and that the lessons in Part First which are marked A. are simply introductory to those which follow, and might be omitted without affecting the completeness of the work.

For the greater simplicity of plan, the work is chiefly confined to Plane Rectilinear Geometry, and treats of this only so far as its laws can be investigated without introducing the doctrine of proportion. The field occupied is very nearly the same with that of the first and second books of Euclid. An index is given of the corresponding propositions both of Euclid and of Legendre.

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* "This identity of direction in all its parts is that peculiar property of the straight line, which enters into every consideration of angles and parallels; and the neglect of which has been the cause of most of the embarrassment that has been felt in discussing the doctrine of parallel lines. . . . . It is hardly credible that the authors themselves, in using parallel lines in the various demonstrations in which they occur, usually think of them as *not meeting*. They contemplate them merely as having the same direction, and mentally derive their results from this property. This is certainly true of those who read their books." — Professor Hayward.

The whole work is so constructed, that it is believed that the teacher, even if the science is entirely new to him, will find no difficulty in taking it up, and, with a little preparatory study, instructing a class in it. Some suggestions in regard to the best method of study and recitation immediately follow the Introduction.

I cannot express strongly enough my obligations to my associate, Professor Chase, for his encouragement, valuable suggestions, and important assistance in the preparation of this work. He has conferred a great additional favor, in consenting to prefix to it an Introduction. I am also deeply indebted to my associate, Professor Young, and to Professor Peirce of the University at Cambridge.

A. C.

March 1, 1847.

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OF THE PROPOSITIONS OF EUCLID AND LEGENDRE, WITH THE SECTIONS OF THE FOLLOWING WORK WHICH ESSENTIALLY CORRESPOND.

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INTRODUCTION.

THE importance of an earlier and more general study of Geometry, the expediency of bringing it into our common schools, and the need of a text-book adapted to that purpose, were, more than seven years ago, subjects of frequent conversation between myself and the author of the following treatise. I was strongly urged by him to undertake the preparation of a book. But no satisfactory plan occurring to me, the subject was dropped. Recently, however, Professor Crosby has again turned his attention to the subject, and, as no work had appeared which met our views, determined himself to make an attempt to supply the want which we had so long felt. The treatise now published is the result of this determination.

I shall avail myself of the opportunity offered by the publication of this work, and of the space afforded me by the author, to suggest some reasons for the introduction of Geometry into common schools, and certain principles, which should direct in the preparation of a text-book for this purpose.

The study of Geometry is admirably *adapted to the powers of the youthful mind.* Its ideas are *elementary.* The ideas of form and size are among the earliest which children acquire. Why should we not continue to direct their attention to ideas so early awakened? Why should we not extend their knowledge of subjects, towards which the mind so

naturally turns, and, while thus encouraging the mind to activity by leading it in the very path which itself has chosen, secure, instead of confused notions, distinct and accurate ideas?

This cannot be difficult. A child can distinguish between a straight line and a curve; can understand the nature of an angle, a triangle, a parallelogram, a square, or a circle; can compare magnitudes, and apprehend the relations of equality and inequality.

In Geometry, moreover, if anywhere, *one thing can be learned at a time*, a principle of admitted importance in the education of the young, though practically far too little regarded.

Again, Geometry is, in many respects, *more elementary and less difficult than Arithmetic.* It is true, that Arithmetic is studied in the child's school; Geometry, in the college. Arithmetic is universally regarded, and spoken of, as an easy study; Geometry is, by some at least, regarded as sufficiently difficult. But the difficulties of Geometry are present or recent difficulties; those of Arithmetic have been forgotten with the griefs of childhood. Arithmetic has been a subject of study from infancy. One of the earliest mental operations which the child learns is to count. Yet it is an operation of no little difficulty. The relations of different numbers are to be learned, and a name to be remembered for each particular number; in other words, for each combination of units. The *number* of the things counted is to be *abstracted* from the *things* themselves. Is not this at least as difficult as to distinguish varieties of form, or to learn the relations of different magnitudes? Is it not as easy to apprehend the distinction between a polygon with three angles, and one with four, as between the abstract numbers 3 and 4? Is it not as easy to see that two equal straight lines will coincide, as to see that 9 times 9 are 81? Or, to understand the definition of a straight line, as to appreciate the local value of figures? Is not, in short, the idea of extension as elementary as that of number? Will a child amuse himself by cutting a sheet of paper into some definite number of pieces, or by endeavouring

to produce some particular shape? by counting the pieces, or by comparing the forms with each other, and with other known forms?

"But why," it will be asked, "is the study of an *elementary* subject frequently so difficult to young men of more mature minds?" Partly from the very fact of the maturity of their minds, or, rather, of their mental habits. They have long thought and spoken loosely of the distinctions of form and magnitude, and have never given a moment's attention to the careful and discriminating consideration of such subjects. Is it strange, that, when young men with such habits undertake to acquire, in a short time, perfectly correct notions of so many objects of study, either entirely new, or seen in new relations, they should find difficulty? Here are serious obstacles to be encountered, aside from any intrinsic difficulty in the subject. First, the subject is new, and every thing connected with it strange. Then, at the age at which they usually commence the study, they cannot, as they might at an earlier period, afford time to stop on each principle as it occurs, and revolve it, and view it on all sides, till they become thoroughly acquainted with it, before looking at another. Principle after principle must be learned in rapid succession; and, if a single principle is passed before it is perfectly known, the mind instantly becomes confused, and the study difficult.

Another obstacle is found in the very *simplicity* of the subject. The topics first considered are so simple, that young men frequently cannot be persuaded to dwell upon them, and give them thought enough to make them perfectly familiar. Thinking it beneath them to sit down to learn the definition of a straight line or an angle, they give but little attention to the first principles, and wait for something more worthy of their study, till they find themselves lost in difficulties resulting from insufficient acquaintance with those very principles which they deemed so insignificant.

Now, at the age when we would have this study commenced, there is ample time to make every term and principle, as it occurs, perfectly familiar, and to illustrate and im-

press it by all the explanation which the subject admits. The pupil, moreover, at this age, will not take offence at the simplicity of the first principles, but will study them the more zealously on account of that simplicity, which brings them within his reach.

Another consideration should be regarded in estimating the difficulties experienced by more advanced students. Suppose young men should enter college, unacquainted with the first principles of Arithmetic, would they find no difficulty in that subject? Or suppose them utterly ignorant of Orthography, and that their college recitations consisted of spelling-lessons to be committed to memory, and accurately delivered in the recitation-room. Should we hear no complaints of the difficulty, the *impossibility*, of learning the lessons? And might it not then be argued with equal force, that acquisitions so difficult for college students certainly ought not to be required of the youth of an academy, and far less of the children of a common school?

Geometry will *interest the young* no less than the more mature. The mind delights to be fully employed upon subjects not beyond its reach. Geometry condescends to the powers of the young, while, at the same time, it furnishes abundant employment to the most mature. Subjects connected with it are among the first that interest the young mind, and the interest need not flag, so long as new truths remain to be considered; in other words, till the boundless resources of Geometry are exhausted.

The celebrated and excellent Pascal was in his early youth purposely restrained by his father from the study of Geometry, lest he should become so much interested in it as to neglect other studies. But the boy could not be prevented from casually hearing the conversations of the mathematicians who frequented his father's house. He heard and was interested; and, when he was about twelve years old, his father found him one day, in his room, alone and busy with a geometrical diagram. He had demonstrated, unaided by any book or teacher, the proposition, that the sum of the angles of a plane triangle is equal to two right angles. The father, notwith-

standing his previous caution, was pleased with the result, removed the interdict upon Geometry, gave him a Euclid, and encouraged him to study it.

A mathematician is said to have hired an individual to study Geometry, paying him wages, as for any other labor. At length, however, he told his pupil, that he could no longer bear the expense, and must forego the service. The pupil replied, that, if the mathematician would continue to teach him, he would willingly study without pay. This arrangement was made, and the study was zealously pursued, until at last the teacher informed his pupil, that he could no longer afford the time to teach him, but must employ it in earning something for his own support. The pupil replied, "If you must earn money, why not earn it as well by teaching me Geometry as in any other way? I will most gladly, not only study without pay, but pay you liberally for teaching me." The pupil was interested, and the study had become a source of enjoyment to him.

Such, substantially, I believe, will always be the result of thorough and well directed study of Geometry. One can hardly entertain clear and exact ideas of the properties and relations of magnitudes without a feeling of delight. And not only is the contemplation of the objects themselves, and their relations, a source of pleasure, but equally so is the consciousness of perfectly clear and accurate knowledge in respect to them, — knowledge free from every shadow of doubt, either as to its truth or its precision.

Again, no study is better adapted than Geometry to *discipline the minds of the young*. It is within their grasp; it interests, excites, tasks, and stores the mind; — not only stores it with useful knowledge, but furnishes it with *valuable habits*. This, which should be the grand object of intellectual discipline, — the formation of good mental habits, — is far too little regarded in our schools. The great effort, too often, is merely to communicate what is called *practical* or *useful knowledge*.

The storing of the mind with facts and principles for future use is indeed important, but it is still more important to

secure habits of right mental activity; habits of accurate perception, of cautious and exact reasoning, of love for truth, of modest self-reliance, of untiring application, and of profound and continued attention. That all these habits are cultivated by the study of Geometry will not be denied. "Pure mathematics," saith Lord Bacon, "do remedy and cure many defects in the wit and faculties intellectual; for, if the wit be dull, they sharpen it; if too wandering, they fix it; if too inherent in the sense, they abstract it." *

The effect of such mental habits, or of the want of them, will be felt in the *studies of the whole course of education.* The principles of Arithmetic should be demonstrated as rigorously as the propositions of Geometry. Unless this is done, Arithmetic is not *learned.* How much better this will be done by one accustomed to geometrical demonstration, needs scarcely a remark.

It is not out of place to remark here, that, in comparing the difficulties of Arithmetic and Geometry, the latter should be compared, not with Arithmetic, learned, as it too often is, by rote, with reference merely to mechanical practice, but with Arithmetic studied as it should be, intelligently and demonstratively. Arithmetic, thus studied as a science, will be found, in general, not less difficult than Geometry, and will, in many cases, more severely task the youthful mind.

The memory, it must be remembered, is not the only faculty to be cultivated. Yet this is, too often, the faculty chiefly developed in our schools, even in teaching Arithmetic. The proper teaching of Geometry will correct this error. A new method of study will be required, other faculties developed, and a change of mental habits effected, which will be most beneficially felt in all other studies, as well as in the whole subsequent life.

The *practical utility* of Geometry is too obvious to need discussion.

Some, however, while they acknowledge the practical utility of Geometry, and its appropriateness and value as a means

* Advancement of Learning. Book II.

of intellectual discipline, will perhaps object, that *the common school is not the proper place for it.*

To this I reply, that it is the very place, and that, too, for several reasons. Can good habits be formed too early? Shall we gain any thing by delaying till young men come into the higher schools and the colleges? till bad habits are fully formed, and confirmed by long practice?

Again, multitudes who never enter the halls of a college, many who never enjoy the instructions even of an academy, whose whole school education is completed in the unpretending way-side school-house, need the knowledge of Geometry in the business of every day of their lives. Carpenters, wheelwrights, workers in tin and in the other metals, mill-wrights, land-surveyors, measurers, engineers, designers, navigators, cannot all be expected to have the advantages of collegiate or of academic education. Must they therefore be deprived of this so valuable discipline, and rest satisfied with the merely mechanical application of geometrical truths, without that knowledge of principles which would contribute so greatly to their own ease and satisfaction, and to the security of the interests intrusted to them?

And not only those who will need to apply the results of geometrical investigation, but *all* who receive their education in the common schools, should have the benefit of this discipline. This remark applies as truly to females as to males. For their intellectual strength, and consequent influence and respectability in society, they need this invigorating discipline. And we can scarce estimate too highly the advantage in educating the next generation, if mothers generally had the benefit of such training, so as to excite their children, by the true intellectual stimulus of sympathy, to the same habits of exact thinking and reasoning.

Still, it may be said by some (few, however, I hope), that it is better not to introduce new studies into the common school, but to confine it strictly to its *proper sphere.* But what is its proper sphere? In very many schools, till within a few years, the almost exclusive objects of attention were Spelling, Reading, Writing, and Arithmetic. English Gram-

mar and Geography, since become so common, were, in many places, unknown. At the same time, the knowledge acquired, even of Arithmetic, was very limited; many, especially females, aiming at nothing beyond the four Simple Rules and perhaps Reduction.

Many then feared, and some still fear, that the introduction of other studies would occasion the neglect of Spelling, and, in the words of a worthy patron of that system, of "the three R's, Reading, Riting, and Rethmetic." But what has been the fact? At the present time, Geography and English Grammar are common studies in all our schools; Arithmetic is universally studied, and, by most, to a far greater extent than formerly. At the same time, many new studies have been introduced, and among them Algebra, a subject certainly not more elementary than Geometry. But do the youth of the present day spell less correctly, and read less fluently and intelligently, than the youth of our schools thirty or fifty years ago? Far otherwise, we believe. In fact, a great part of the time then spent in school was wasted, for want of something to awaken interest, — *something to do.* Almost the whole time of the school was occupied in reading and spelling, and that in no very intelligent way, as is testified by the unnatural, monotonous voice, and the absence of all conversational tones, so often observed. Spelling, then, we think, — reading, we know, — is better taught now than formerly; and all the additional studies are clear additional gain.

Why then stop where we are? Our schools are longer than they then were, the children more at leisure to attend them. Will it not be still an additional and important gain, if we can secure in them the commencement of another science of so great utility as Geometry?

It is interesting to observe the progress made in the knowledge of Geometry, and in its diffusion through various classes of the community, during a period of twenty-five hundred years. We first hear of it, as an occult and mysterious science, among the Egyptian priests. We next find it in Greece, among the philosophers and learned men; — cultivated by Thales, Pythagoras, Plato, and their schools, and by Euclid,

Archimedes, and Apollonius. Thales is said to have brought it into Greece, about six hundred years before the Christian era; he is also said, which indicates the state of the science at that time, to have himself discovered that the angle inscribed in a semicircle is a right angle, and to have testified his joy by a sacrifice to the Muses! Pythagoras and Plato also are said to have extended the bounds of geometrical knowledge; the former, who lived about 550 B. C., being understood to have discovered that elegant proposition, which still bears his name, respecting the squares described on the sides of a right-angled triangle. He also is said to have expressed his joy and gratitude to the gods, by the sacrifice of a hundred oxen. Whatever may have been the facts in regard to these discoveries and sacrifices, the manner of their mention sufficiently indicates the limited extent of the science at that time, even among the greatest philosophers. Euclid, who lived about 280 B. C., has left us more abundant evidence of the state of the science in his time, in his "Elements of Geometry," a work which held its ground as the principal, almost the only, text-book on the subject for more than two thousand years; until his name became a synonyme for the science, and men spoke of studying, not Geometry, but Euclid. Later still, Archimedes and Apollonius distinguished themselves in the higher departments of mathematical science; Apollonius, particularly, by a most valuable treatise on the Conic Sections. Archimedes, "the most profound and inventive genius of antiquity," is celebrated, not only for his mathematical science, but for his mechanical skill, by which he defended Syracuse, for a considerable time, against the utmost exertions of a Roman army, and for the boast, that, if he had a place on which to fix his lever, he would move the world.*

Now, during all this time, and for many centuries after, the knowledge of Geometry was confined to the philosophers, —to the few. Out of the schools of philosophy, among the mass of the community, such science was utterly unknown. The universal diffusion of knowledge, as of all other blessings,

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* Δὸς ποῦ στῶ, καὶ τὸν κόσμον κινήσω.

is the suggestion of Christianity. The properties of the circle and the triangle, at whose discovery Thales and Pythagoras are said to have been so elated, are now known to the tyro. Pascal, at the age of sixteen, composed a treatise on the Conic Sections, in which he gave, in a single proposition and four hundred corollaries, all that had come down from Apollonius, "the Great Geometer" of antiquity. And what one boy of sixteen may write, another boy of sixteen may learn. We believe that Geometry, instead of being confined, as formerly, to philosophers, or, as more recently, to an educated class, will, at a day not far distant, be introduced into the common schools all over our country, and brought within the reach of every boy and girl in the community.

This is the *proper sphere* of the common school; not to communicate a fixed amount of knowledge, the same to our children as to our fathers, but to communicate continually additional knowledge, and to produce higher and higher degrees of intelligence; — when men of science extend the bounds of knowledge, to diffuse that knowledge, till the world enjoys its benefits. This *universal and progressive diffusion of knowledge* constitutes the *proper sphere* of the common school.

But if Geometry is to be studied in our common schools, in what shape shall it be presented? What principles should direct in the preparation of a text-book for this purpose?

1. The *definitions* should be *perfectly clear* and *exact*. A definition should equally avoid excess and defect. It should express neither too much nor too little. It should be so full as perfectly to identify the object defined, but should not include properties the possibility of whose combination is yet to be proved.

Thus, Legendre's definition of the circumference of a circle as "a curved line, all the points of which are equally distant from an interior point, called the centre," * does not

* "La *circonference du cercle* est une ligne courbe, dont tous les points sont également distants d'un point intérieur qu'on appelle *centre*.

identify the circumference of a circle; but is equally applicable to any line whatever drawn upon the surface of a sphere. It should be defined, "a curved line *in a plane*, all the points, &c."

Again, when, before any demonstration of the properties of quadrilaterals, the square is defined as "a quadrilateral, which has all its sides equal, and all its angles right angles"; the rectangle, as one "which has its opposite sides equal, and its angles right angles"; the parallelogram, as one "which has its opposite sides and angles equal," or "its opposite sides equal and parallel," — these are examples of excess in definition. How do we yet know that a quadrilateral *can* have, at the same time, its opposite sides equal and parallel, or its opposite sides and angles equal, or its opposite sides equal and its angles right angles, or all its sides equal, and its angles right?

Clearness and simplicity in the definitions are promoted by a natural *order of succession*. Of the quadrilaterals, for example, the square is, in many of the books, defined before the rectangle, and the rectangle before the parallelogram, that is, the *species* before its *genus*. Whereas the true order of science requires us to proceed from the more to the less general, adding, at each step, only the necessary limitations or specifications.

An object should be defined by means of that *distinguishing property*, from which its other properties may be most easily and satisfactorily deduced. The fact that *parallels* never meet seems to be less their distinguishing property, than a consequence of some other property. This property furnishes no convenient means of drawing parallels, nor any practical test of parallelism. It is much more satisfactory and convenient to define them as *having the same direction*.

2. The *propositions* should be *enunciated with the utmost precision*. A defect in this respect sometimes amounts to a gross error. Thus, in some of the books, we find this prop-

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"Le *cercle* est l'espace terminé par cette ligne courbe." — *Éléments de Géométrie*, 11[me] Éd., p. 33.

osition. "If the product of two quantities be equal to the product of two other quantities, two of them" (any two, of course) "may be made the extremes, and the other two the means, of a proportion"; e. g. $4 \times 8 = 2 \times 16$; then, making 4 and 2 the extremes, we have $4 : 8 = 16 : 2$; or, as the product of the extremes is equal to that of the means, $4 \times 2 = 8 \times 16$, or $8 = 128$. It should be, the *two factors of one product* may be made the extremes, &c. Nor is this a mere captious objection. I have repeatedly known students to make the mistake, and find it out only by trying to verify the result.

Again, a good enunciation distinguishes, and marks the distinction with great care, between *hypothesis* and *conclusion.*

3. The *most rigorous exactness of demonstration* must be preserved. We want no tentative or experimental methods of proof. Empiricism is as bad in mathematics as in medicine. We want no practical results to be learned by rote, without proof. We must have proof, — infallible proof, — demonstration. The reasoning may be simplified, and reduced to the comprehension of the young, by multiplying and shortening the steps, if need be; but still *it must be demonstration.*

4. The memory is aided, fresh interest awakened, and the whole mind invigorated, by the *generalization of geometrical truths;* a process which connects under one enunciation several apparently distinct propositions, and shows them to be only particular applications of a more general principle, — specific forms of a generic truth.

5. A book designed for elementary schools should abound in *minute and familiar illustration,* both of terms and principles. The definitions and propositions should be expressed as concisely as possible, so that they may be easily remembered, and conveniently quoted. But the terms used should be explained with great care, and such remarks added as will connect and show the relation between the rigorous expressions of Geometry, and the looser language of ordinary conversation.

The practical application of principles should be set forth, not to strengthen the proof of a proposition, — for the infallible nature of demonstration, and the impossibility of increasing its certainty by additional evidence should be constantly insisted on, — but rather to illustrate the meaning of the abstract principle, to show how readily it applies itself to practical results, and connects itself with common things, and so, at the same time, to aid, to interest, and to benefit the pupil.

It may be thought, that the teacher should supply the necessary illustrations. He should, indeed, so far as his time permits; but yet, for various reasons, besides the want of time, the burden should not be wholly thrown upon him.

In the first place, many will be called upon to teach Geometry who are not particularly interested in it; some, perhaps, who are not very familiar with it; and some, possibly, like a teacher I once knew, who thought that "Euclid and English Grammar could be learned only by committing them to memory." To such teachers, illustrations, unless suggested by the book, will not be likely to occur.

Another difficulty will be experienced by the most accomplished teachers. Explanations, if first suggested during the recitation, will not generally be appreciated or remembered. The pupil should *study them with his lesson;* he should reflect upon them, and see for himself their conneetion with the subject. He will then be prepared to appreciate any additional remarks from his teacher, and to ask intelligent questions of his own. The more abundantly illustrations are furnished by the book, the more readily will additional illustrations occur to both teacher and pupils.

It may not be out of place here to remark, that a difficulty, of far greater magnitude, indeed, but of the same nature, is occasioned by *faulty and inelegant definitions and propositions.* If the fault be pointed out before the lesson have been learned, the subject is strange, and the correction not understood or remembered; if afterwards, the faulty expressions will then have been fixed in the mind, and cannot easily be eradicated.

"But after all this simplification, illustration, and improve-

ment of the form and arrangement of the definitions and propositions, 'there is no royal road to Geometry.'" True, the subject cannot be mastered without labor, but that labor may be intelligent and well directed. The height must be scaled, but it need not be approached on its most precipitous side. The road must pass over the loftiest summit, but, by beginning early, the ascent may be gradual and easy.

S. C.

Dartmouth College, Jan. 22, 1847.

SUGGESTIONS

IN REGARD TO THE STUDY OF THE FOLLOWING WORK.

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1. In preparing this work, great pains have been taken to make the questions such that the student may be able to answer them all for himself. If, therefore, he finds in a question any words which he does not fully understand, let him ask his teacher their meaning; but, beyond this, let him not ask any assistance either from his teacher or from any one else. "*What man has done, man may do.*" Let him feel, that one of the great objects of studying Geometry is to lead him *to think, and discover, and know for himself.* And let him leave no question, till he is able to answer it, both *knowing that he is right*, and *knowing why he is right.* Geometry is no subject for guess or uncertainty.

2. Let the student learn the DEFINITIONS, AXIOMS, and THEOREMS, not only so as *to understand them perfectly*, but so, also, as to be able *to repeat them in their precise words.*

3. Let the pupil, when called up for recitation, leave his book at his desk; or, if, for any reason, his teacher wishes him to bring it, let him never open it during

the recitation, except by the express direction of his teacher.

4. In reciting the demonstrations, let the pupil never wait for the teacher to ask him questions. Let him go to the blackboard, and draw the diagram, placing upon it *different letters from those given in the book*, and sometimes using the figures 1, 2, 3, &c., instead of letters. Then let him state distinctly *what is given*, and *what is required*. Let him then, without any delay or hesitation, show, *in his own language*, how he obtains the result, proving every step by referring to definitions, axioms, or previous theorems, and repeating these whenever desired. Let him first state this result using the letters of the diagram, and afterwards generally in the words of the theorem. After this, let the teacher ask such questions, or make such remarks, as may serve to test or extend the pupil's knowledge, or increase his interest in the subject. The pupil should be encouraged to find out, wherever he can, different modes of demonstration. Some of these are already pointed out, more or less minutely, in the book itself. It is of great importance that the student should early feel that every path of right reasoning leads to the same result.

5. It is often useful, during recitation, to direct the pupil to write upon the blackboard the successive steps of a demonstration, in the same manner as he would the solution of a question in Arithmetic or Algebra. At other times he may be directed to bring the demonstration from his desk, written out upon a slate or piece of paper. In this exercise, as in oral recitation, the interrogative form which is employed in the book must, of course, be changed into the corresponding positive

form. For example, the demonstration in § 33 would be thus written:

$$AEC + AED = 2\llcorner.$$

And $$DEB + AED = 2\llcorner.$$

$\therefore$ $$AEC + AED = DEB + AED.$$

$\therefore$ $$AEC = DEB.$$

6. It is perhaps needless to remark, that many of the questions are designed to lead the mind of the student to the perception of geometrical truth, rather than to be repeated by the teacher at recitation.

7. Many truths in Geometry are usefully illustrated, especially to the young, by cutting out the figures in paper, and applying them to each other. Let the student, wherever he can, do this for himself.

8. The PROBLEMS are placed, for convenience, in separate parts of the work; but may be taken up, if the teacher thinks it best, in connection with the theorems upon which they are founded. The pupil should carefully perform them upon paper at his desk, and also recite them at the board with their proofs.

9. If a blackboard cannot conveniently be used in recitation, let the pupil recite from the diagrams at the end of the book, where, it will be observed, the letters are somewhat changed. Let him NEVER recite from the diagram upon the same page with the text. A convenient chart for recitation may be made by cutting out the pages of diagrams and pasting them upon a sheet of pasteboard.

10. Finally, let no one who studies these Lessons despise that which seems to him simple; it is a necessary preparation for something higher. Nor let him be discouraged by any thing that may at first appear difficult. Great pains have been taken to adapt every

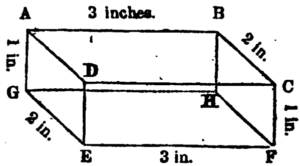
thing to his powers; let him think what others have accomplished, and take as his motto, "WHAT MAN HAS DONE, MAN MAY DO." I wish that every leaf of this book had upon it the same stamp with the paper upon which I am writing these suggestions. It is a hand pointing upward to a scroll, upon which are inscribed only three letters, but those full of meaning, — "TRY."

LESSONS IN GEOMETRY.

PART FIRST.

I. A.

Fig. I.



a.) Take this block of wood (represented above), and tell me if it has any *length*. How long is it?

Has it any *breadth*, or *width*? How broad, or wide, is it?

Has it any *thickness*, or *height*? How thick, or high, is it?

LENGTH, BREADTH, *and* THICKNESS (or height) *are called* DIMENSIONS. — How many dimensions has the block? What are they?

Can you find a block of wood which has not length, breadth, and thickness? Can you find a piece of any other matter which has not?

How many dimensions, then, has every block, or piece of matter?

b.) Has this room any length? any width? any height? How many dimensions then has it?

What is this room? ANSWER. A *portion of space*, inclosed within the sides, ceiling, and floor.

Has the space in that box any length? any breadth?

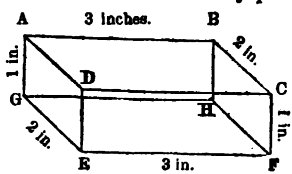
any height (or *depth*, which is the same dimension, measured *down*)?

How many dimensions has the space in a drawer? in a cellar? Can you find or think of any portion of space which has not these three dimensions?

c.) *That which has* LENGTH, BREADTH, *and* THICKNESS *is called a* SOLID. — In Geometry, whatever has the three dimensions of length, breadth (or width), and thickness (or height), is called a solid, whether it be hard or soft, whether matter or mere space.

Is this block a solid? this book? Can you find any piece of matter which is not a solid?

According to the definition, is this room a solid? the space in a box or drawer? every portion of space?



d.) How many FACES OR SURFACES has this block? Which is the upper surface? which, the under? Which are the largest surfaces? which, the smallest?

Take the upper surface (which is marked above by the letters A, B, C, D at the corners), and tell me if it has any length. How long is it?

Has it any breadth? How broad is it?

Has it any thickness? Is the surface all *upon the outside*? Does it then go down into the wood at all? *Can* it then have any thickness?

How many dimensions then has it? Which dimension does it want?

How long is the surface marked by the letters C, D, E, F at the corners? how broad? how thick?

How long is the surface marked ADEG? how broad? how thick?

Can any surface, as it is nothing but *outside*, have any thickness?

How many dimensions then has a surface? What are they? Which does it want?

If you define a SOLID, as above (c), *that which has*, &c.; how would you define a SURFACE? ANS. *That which has* —— *and* ——, *without* —— (in answering, fill up the blanks with the proper words).

e.) How many EDGES has the upper surface of the block (represented above by ABCD), or, in other words, how many LINES are there round it? Point them out, and likewise name them by the letters at the corners, as the line AB, the line BC, &c.

Has the edge, or boundary-line AB, any length? How long is it?

Has it any breadth, or width? Does it all lie *at the very outside* of the surface? Can it then come into the surface at all? *Can* it then have any breadth?

As it is merely the boundary-line of a *surface*, can it have any thickness?

How many dimensions then has it? which? Which dimensions does it want?

Has the line AD any length? any breadth? any thickness?

How long is the line BC? how broad? how thick?

Can any boundary-line of a surface have any breadth or thickness? — The only lines spoken of in Geometry are boundary-lines of surfaces, since a division-line is only a boundary-line between two surfaces.

What dimension then has a line? What dimensions does it want?

If you define a SOLID, *that which has*, &c. (c); and a SURFACE, *that which has*, &c. (d); how would you define a LINE? ANS. *That which has* ——, *without* —— *or* ——.

f.) How many ENDS, or END-POINTS, has the line AB? — These may be called, from the letters near them, the point A, and the point B.

As the point A is merely the end of a *line*, can it have any breadth? any thickness?

Has it any length? As it is only the *end* of the line, does it extend into the line at all? *Can* it then have any length?

Has it then any dimension?

What property then has it? ANS. Nothing but *position*, that is, a place at the end of the line.

Has the point B any length, breadth, or thickness?

Can the end-point of any line have any dimension? — The only points spoken of in Geometry are the end-points of lines, since a division-point is only the common end-point of two lines.

What is then the only property which a point has? What are the three dimensions which it wants?

How then would you define a POINT? ANS. *That which has* ——, *but neither* ——, ——, *nor* ——.

I. B.

§ 1. GEOMETRY[a] is the science[b] of EXTENSION[c] and DIRECTION[d].

(a) From the Greek Γεωμετρία, compounded of γῆ, *land*, and μετρέω, *to measure*. The science was so called from its early application to the *measuring of land*, especially among the Egyptians, to whom it was peculiarly important as a means of ascertaining their boundaries after the inundations of the Nile. See Herodotus, ii. 109. (b) From the Latin scientia, *knowledge*. (c) Lat. extensio, from extendo, *to stretch out*. (d) L. directio, fro-

REMARKS. *a.* Geometry, as a science, first *defines*[e] the objects of which it treats; and states or implies certain *self-evident truths* in respect to them. These truths are termed AXIOMS[f], and are the primary laws of the reasoning which it employs. From these definitions and axioms, it then, by a method of strict proof, or *demonstration*[g], deduces THEOREMS[h], that is, *general truths or laws established by proof*. It lastly shows the application of its truths to the *solution or performance of* PROBLEMS[i].

b. The DEFINITIONS, AXIOMS, and THEOREMS constitute the *theoretical* part of the science; and the PROBLEMS, the *practical*. The theorems and problems, from the usual method of presenting them, have been classed together under the general head of PROPOSITIONS[k]; the theorem *proposing something to be proved*, and the problem, *something to be performed*. A LEMMA[l] is a *theorem introductory to another theorem, or to a problem*. A COROLLARY[m] is an *inference from that which precedes*. A SCHOLIUM[n] is an *accompanying remark*.

c. In every proposition, it is essential to distinguish accurately between *that which is given*, and *that which is required*. *Things given* are called DATA[o]; and the *data of a proposition* are termed its HYPOTHESIS[p]. A *supposition not included in the data* is termed an ASSUMPTION[q].

d. EXTENSION and DIRECTION are *simple ideas*, and are *implied in every conception of space*[r], *or of bodies occupying space*. They are first suggested to the mind by material objects, and these objects afterwards assist the mind in the

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dirĭgo, *to direct.* (e) Lat. definio, *to limit.* (f) Gr. ἀξίωμα, from ἀξιόω, *to deem worthy, suppose, take for granted.* (g) L. demonstratio, from demonstro, *to point out, prove.* (h) Gr. θεώρημα, from θεωρέω, *to view, contemplate.* (i) Gr. πρόβλημα, from προβάλλω, *to throw* or *lay before.* (k) L. propositio, from propōno, *to place before, to propose.* (l) Gr. λῆμμα, *premise*, from λαμβάνω, *to take.* (m) L. corollarium, *something given over and above*, from corolla, *a wreath*, a common present or mark of honor. (n) Gr. σχόλιον, *comment*, from σχολή, *leisure*, in a special sense, *leisure devoted to learning, school.* (o) L., from do, *to give.* (p) Gr. ὑπόθεσις, *foundation*, from ὑποτίθημι, *to place beneath, lay down.* (q) L. assumptio, from assūmo, *to assume.* (r) L. spatium.

investigation of their laws; still it must be kept distinctly in view, that they are alike the properties of space, whether *occupied* or *unoccupied* by matter; and that they belong to matter simply from its occupying space.

e. EXTENSION and DIRECTION become objects of science as *admitting comparison and measurement*, or, in other words, as possessing DIMENSION[s]. The general term MAGNITUDE[t] is applied to *whatever can be measured*. Hence there are two kinds of magnitudes treated of in Geometry, those which may be referred to extension, and those which may be referred to direction; and we may use the term alike, whether treating of pure space, or of bodies occupying space.

f. In geometrical statement and reasoning, it is convenient to employ figures addressed to the eye, termed DIAGRAMS[u], and to designate the magnitudes of which we are speaking by *letters* or *other marks* placed upon these diagrams. But it must be distinctly understood, that, in the language employed, we are not speaking of the diagrams themselves, but *universally of all such magnitudes, whether material or of pure space, as agree with the data;* and that the diagrams are merely employed to represent these magnitudes to the eye, and to furnish a convenient language for speaking of them. The reasoning of Geometry is not *inductive*, from particulars to generals; but *deductive* or *demonstrative*, from generals to other generals; and it were as reasonable to suppose that the geographer, in his descriptions, is speaking of his maps, or the biographer, in his narrative, of the portrait prefixed to his volume, as that the geometer, in his reasonings, is treating of his diagrams.

§ 2. EXTENSION has three dimensions; LENGTH, BREADTH, and THICKNESS.

REMARKS. *a*. By *one measurement*, we obtain LENGTH; by a *second*, *across the first*, BREADTH (or WIDTH); by a *third*, *through the other two* (as *up* or *down*, if the other two

(s) Lat. dimensio, from dimetior, *to measure*. (t) L. magnitūdo, from magnus, *great*, *large*. (u) Gr. διάγραμμα, *drawing*, from διαγράφω, *to draw out*, *delineate*.

are horizontal), THICKNESS (or HEIGHT). From the very nature of space, no fourth dimension is possible.

b. In measurement, the *first* dimension obtained or contemplated is termed length; the *second*, breadth; and the *third*, thickness. Where there are two or three dimensions, it is evident that the order of measurement, and consequently the application of the terms, may be varied.

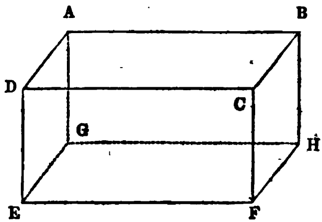
§ 3. The MAGNITUDES OF EXTENSION are of three kinds, according to the number of their dimensions; SOLIDS[v], which have *three dimensions;* SURFACES[w], which have *two;* and LINES[x], which have only *one*.

REMARKS. *a*. All portions of space, and all bodies occupying space, have, of necessity, length, breadth, and thickness, and are consequently solids. Surfaces are the mere *outsides* of solids; and consequently, while they have length and breadth, can have no thickness. Give to a surface any thickness, and it is no longer the mere outside of a solid, but itself a solid. Lines are the mere *limits* or *edges* of surfaces; and consequently can have only length, without breadth or thickness. Give to a line breadth, and it is no longer the mere limit of a surface, but itself a surface.

b. The *extremities of lines* are termed POINTS[y]. These have *position*, but can have no dimension. Give to a point length, and it is no longer the mere end of a line, but itself a line.

c. Thus *solids are bounded by surfaces; surfaces, by lines; and lines, by points*. Even a surface dividing a solid is only the common boundary of two solids; a line dividing a surface, of two surfaces; and a point dividing a line, of two lines. Surfaces, lines, and points exist as really as solids. But they exist only *in* solids, as *properties* of those solids. Still there is nothing to forbid our conceiving and speaking

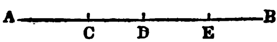
(v) Lat. solidus. (w) French, from sur, *upon*, and face, *face*, that which is upon the face or outside; so, in Lat., superficies, from super and facies. (x) L. linea. (y) Fr. point, Old Fr. poinct, from Lat. punctum.



of them *abstractly*. The solid AF, represented in the margin, is bounded by the six surfaces AC, CE, EH, HA, AE, and BF. The surface AC is bounded by the four lines AB, BC, CD, and DA. The line AB is bounded by the two points A and B.

d. From the idea of extension arises that of *distance*[z]; and by combining distance and direction, we obtain that of *position*[a]. The ideas of extension, and of direction or position, produce that of *form*[b]. Change of position constitutes *motion*[c]; which, if not an essential idea in Geometry, is of great importance as an auxiliary. Thus, each of the magnitudes of extension may be regarded as produced by the motion of its limit; a line, by the motion of a point; a surface, by the motion of a line; and a solid, by the motion of a surface.

II. A.



a.) Draw, between the points A and B, a line of which the direction shall be everywhere the same; that is, so that, if any number of points, as C, D, E, be taken in the line, the direction from A to C shall be the same with that from A to B; the direction from C to D, the same; and in like manner that from D to E, from E to B, from A to D, from C to E, &c.; or, reckoning from B, so that the direction from B to E, from E to D, from D to C, from C to A, from B to D, &c., shall be the same with that from B to A.

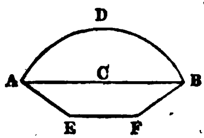
(z) Lat. distantia, from disto, *to stand apart from.* (a) L. positio, from pono, *to place.* (b) L. forma. (c) L. motio, from moveo, *to move.*

A line, of which the direction is everywhere the same, is called a STRAIGHT LINE. — Is AC a straight line? Is AD a straight line? DB? CB?

Can a straight line have any *bend* or *crook* in it?

b.) Draw, from A to B, the straight line ACB. Draw, below it, a second line AEFB, bent at E and F, but straight from A to E, from E to F, and from F to B. How many bends has the line? Of how many straight lines is it composed? Have these straight lines the same direction? How many times does the direction change?

II.



A bent line composed of straight lines is called a BROKEN LINE. — What kind of a line is AE? EF? AEF? FB? EFB? ACB? AEFB?

Draw a broken line with 3 bends; with 5; with 7. Of how many straight lines is each composed?

How many times does the direction change in a broken line composed of 4 straight lines? of 5? of 7? of 12?

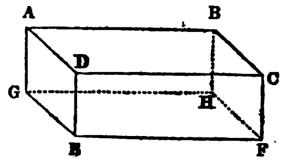
c.) Draw from A to B, in the figure above, a line, as ADB, bent at every point. Is any part of the line straight? Does it change its direction at every point? — *A line of which no part is straight* is called a CURVED LINE, or a CURVE.

Draw, from A to B, another curve outside of ADB; and another still outside of this.

If these lines are all curves, do they bend at every point? Do they all bend *alike*? Have they the same, or different forms?

Can curves differ in form? Can broken lines? Can straight lines?

d.) In the surface AC of the block AF,* do you see any bend ? If it has no bend, is the direction of the surface everywhere the same ? — *A surface of which the direction is everywhere the same* is called a PLANE SURFACE, or a PLANE. — Does AC appear to be a plane ? DF ? AE ?



Is every perfectly flat surface a plane ?

e.) Is the direction of the surface AC, above, the same with that of DF ? Taken together, do they make a plane or a bent surface ? In a perfectly round ball or marble, is any part plane ?

A bent surface composed of planes is called a BROKEN SURFACE ; *one of which no part is plane*, a CURVED SURFACE. — What kind of a surface has a globe ? the roof of a house ?

Mention bodies which have plane surfaces ; broken surfaces ; curved surfaces.

Take a piece of paper, and hold it so as to represent a plane surface. Bend it so as to represent a curved surface. Bend it so as to represent a broken surface.

II. B.

§ 4. DIRECTION belongs to *points*, *lines*, and *surfaces*. The direction of *lines* is termed LINEAR[d]; of *surfaces*, SUPERFICIAL[e].

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* Surfaces and solids are often named by simply taking two letters at opposite points.

(d) Lat. lineāris. (e) L. superficiālis. See Note w, page 37.

REMARKS. *a.* When the direction of a *solid* is spoken of, the direction of some *point, line,* or *surface in the solid* is properly meant.

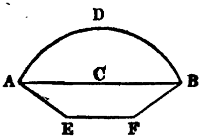
b. Points can have direction only with reference to other points; and, in measuring this direction, lines are supposed to be drawn. Hence, the direction of points may be practically referred to linear direction.

c. The important distinction between *superficial* and *linear* direction must be kept carefully in view. Thus, a surface may lie in a horizontal direction; and then any line drawn in the surface will be horizontal. But this line may be drawn east, west, north, south, or to any other point of the compass; so that *a line both partakes of the direction of the surface in which it lies, and has also its own particular direction in that surface.*

d. In a line or surface, we may consider either *the direction of the parts in reference to each other* (§ 5), or *the direction of the whole in reference to another line or surface* (§ 7).

§ 5. A LINE is termed STRAIGHT, and a SURFACE, PLANE[f], if *the direction is everywhere the same*; but *otherwise*, BENT. A BENT LINE, or SURFACE, is termed BROKEN, if *composed of straight lines*, or *plane surfaces*; but CURVED[g], if *no part* of the line is *straight*, or of the surface *plane*.

II.



Thus ACB represents a straight line; AEFB, a broken line; and ADB, a curved line, or, as it is commonly termed, a *curve*. The surfaces represented in the figure in § 3. *c* are plane surfaces, or, as they are commonly termed, *planes*.

(f) Lat. planus, *flat, even.* (g) L. curvus, *crooked, bowed.*

REMARKS. *a.* A bent line or surface is termed *convex*[h] towards the side which the middle approaches, and *concave*[i] towards that from which it recedes. Thus ADB is convex above, and concave below, and AEFB the reverse. These terms are applied chiefly to curved lines and surfaces.

b. Into how many soever smaller lines a straight line is divided, or of how many soever it is composed, or how far soever it is extended either way, still if the direction is nowhere changed it is considered as throughout one and the same straight line. A similar remark may be made of the plane.

c. A straight line, or plane, is said to be *produced*[k], when it is extended in the same direction. To *join two points* is to draw a straight line from one to the other. The expression, *to join* AB (or, *to draw* AB), means, to draw the straight line AB joining the two points A and B.

d. A PLANE FIGURE is one every part of which lies in the same plane. It is *rectilinear*[l], if all its lines are straight; *curvilinear*, if any of them are curved. PLANE GEOMETRY is that portion of the science which treats of plane figures. It is either RECTILINEAR or CURVILINEAR, according to the lines introduced. In the earlier part of Geometry, it is understood that all the figures introduced are plane figures, unless the contrary is expressly stated. The simple term *line* is also understood to denote a *straight line*, unless the connection obviously requires a different sense.

e. Lines, surfaces, and figures, in which curves are joined with straight lines, or curved surfaces with planes, are sometimes termed *mixt*[m].

f. In the definitions of a straight line and plane, the distinction between *superficial* and *linear* direction must be kept in view (§ 4. *c*). Thus, in a PLANE (regarded as lying horizontally), it is simply required, that no part *rise above* or *sink below* a uniform direction; while in a STRAIGHT LINE

(h) Lat. convexus, *arching out.* (i) L. concăvus, *hollow.* (k) L. prodūco, *to draw out, prolong.* (l) L. rectus, *right, straight,* and linea, *line.* (m) L. mixtus, *mingled.*

lying in that plane, it is required, that no part either *rise above*, or *sink below*, or *bend sideways from*, a uniform direction.

g. It is beyond the power of human art to make either a perfect straight line, or a perfect plane. But Geometry admits no imperfection; and all the objects of which it treats must be conceived of by the mind as *absolutely perfect*, although they cannot be so represented to the eye.

III. A.

a.) In the straight line AB, if you reckon from A, is the direction everywhere that from A to B? If you reckon from B, what is it?

So far as the line itself is concerned, does it make any difference which way you reckon?

Does it make any difference in the *road itself* which way a traveller is going? May it make some difference to *him*? How many ways can a traveller go upon the same road?

How many ways may the direction of any straight line be reckoned?

b.) Now try to draw a second straight line from A to B. If the line is straight, what must its direction everywhere be?

But what was the direction of the first line, reckoning from A?

Can the second then anywhere leave the track of the first? Can it make, then, a distinct, or separate, line, or must it be the same with the first?

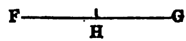
Try to draw a second straight line from B to A. What must its direction everywhere be ?

But what was the direction of the first line, reckoning from B ? Can the second then anywhere leave the track of the first ? Can it then make a distinct line, or must it be the same with the first ?

How many distinct straight lines then can be drawn from A to B or from B to A, or, in other words, can lie between A and B ?

How many can lie between A and D ? between C and E ? between any two points ?

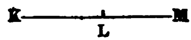
c.) If a straight line be drawn from F towards G, and one at the same time from G towards F, will they meet, if drawn far enough ?



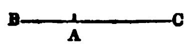
If they should not meet, but should pass by each other, how many distinct straight lines should we then have between F and G ? Would this be possible ?

If they meet at H, will they form one straight line ?

Let KL and LM be straight lines having the same direction and meeting at L, what do they form ?



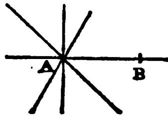
d.) Through the point A, draw the straight line BC. Now try to draw through A a second straight line having the same direction with BC.



Whether you draw this line towards C or towards B, can it anywhere leave the track of the first ? Can it then make a distinct line ?

Then, how many distinct straight lines having the same direction can pass through the point A ? through any one point ?

e.) If, in a straight line, you know the position of one point, A, do you know the *direction* of the line? How many straight lines can you draw through the point A? through any one point?



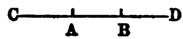
But, if you also know the position of a second point, B, do you then know the direction of the line?

What must be the direction of a line of which A and B are two points?

Do you also know the *position* of the line (that is, its place, without regard to its length)?

How many distinct straight lines can each pass through both A and B?

f.) If two straight lines, CB and AD, have the two points A and B common (that is, lying in each of the lines), can CB and AD be distinct straight lines, or must they form one and the same straight line? Why?



If two straight lines, CB and AD, have a part AB common, are they distinct straight lines, or one and the same?

How many points can two distinct straight lines have in common?

In how many points can two straight lines *intersect* (that is, cut, or cross, each other)?

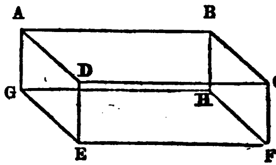
If two straight lines *meet* at any point, can they meet at any other point?

Can two straight lines then surround, or inclose, a figure?

g.) *Produce* AB to C (§ 5. *c*). What kind of a line is ABC ? Produce BA to D. What kind of a line is DAB ? DABC ?

Join AB (§ 5. *c*). What kind of a line is AB ?

h.) In a plane, as ABCD, take any two points. Is the direction of the plane everywhere the same between these two points ?



Join the two points by a straight line. Is the direction of this line everywhere the same ?

Can it then anywhere rise above or sink below the plane ? Must it lie wholly in the plane ?

If, upon a perfect plane, you lay a line which is perfectly straight, will it touch the plane throughout, or not ?

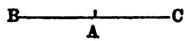
III. B.

§ 6. It is evident from the definition of a straight line, (*a*) that ***two distinct straight lines cannot lie between the same two points*** (that is, extend from one point to the other) ; (*b*) that ***two distinct straight lines having the same direction cannot pass through the same point*** ; (*c*) that, ***if two straight lines have two points*** (or, in other words, ***any part***) ***common, they must form one and the same straight line*** ; and, therefore, (*d*) that ***any two points in a straight line determine its direction and position*** ; (*e*) that ***two straight lines can intersect***

or meet in only one point; and, consequently, (*f*) that *two straight lines cannot inclose a figure*. It is also evident from the definitions of a straight line and of a plane, (*g*) that, *if any two points are taken in a plane, the straight line joining them must lie wholly in the plane*.

REMARKS. 1. The mode of testing a plane, by applying to it a straight edge in various directions, is explained by *g* above.

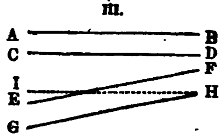
2. The direction of a straight line is twofold in its nature. Thus the direction of AB is the direction from A to B, or, reversely, the direction from B to A. So far as the single line itself is concerned, it makes no difference which way its direction is reckoned, whether from A to B, or from B to A. But in its relations to other lines, the distinction becomes of essential importance; and, in this view, every straight line may be said to have two directions, the one the reverse of the other. If a straight line is divided by a point, and the direction of each part is reckoned from that point, then one part may be termed the *reverse*[n] of the other. Thus, if BC is divided at A, and the direction of AC is reckoned from A to C, while that of AB is reckoned from A to B, then AB is the reverse of AC, and, on the other hand, AC is the reverse of AB.



IV. A.

a.) Draw the straight line AB. Below it draw CD, having the same direction with AB. Below this draw EF, having a different direction

III.

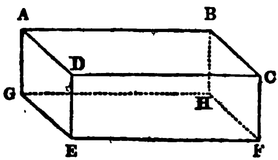


(n) Lat. reversus, *turned back*.

from CD; and also GH, having the same direction with EF.

Straight lines, or planes, are said to be PARALLEL *to each other, when they have the same direction;* INCLINED, *when they differ in direction.* — Are AB and CD parallel to each other or inclined? CD and EF? AB and EF? EF and GH? CD and GH? AB and GH?

b.) In the block AF, do the planes ABCD and GHFE appear to have the same or different directions? Are they then parallel to each other or inclined?



Are AC and DF parallel to each other or inclined? DF and AH? AH and EH? AE and BF?

How many pairs of parallel planes does the block appear to have?

c.) Draw IH parallel to CD (Fig. III.). Has it the same direction with CD?

Has AB the same direction with CD?

Have then IH and AB the same direction? Are they parallel?

If any two lines are both parallel to a third line, how must they be related to each other?

If two *planes* are both parallel to a third plane, how must they be related to each other?

If one of two parallel lines, or planes, is *inclined* with respect to a third line, or plane, what would you say of the other?

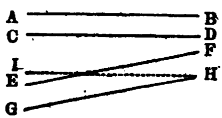
d.) How many straight lines having the same direction can pass through the same point (§ 6. *b*)?

Can parallel lines then ever meet?

IV. B.

§ 7. **Straight lines, or planes,** are said to be **parallel**[o] to each other, when they *have the same direction*; **inclined**[p], when they *differ in direction*.

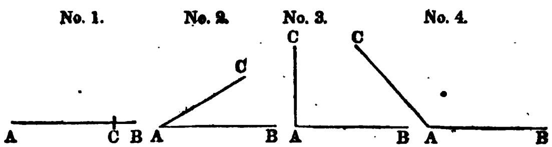
AB and CD represent parallel lines (or, as they are often termed, *parallels*). EF and GH represent lines which are parallel to each other, but are inclined with respect to the other two. In the figure in § 3. *c*, DF and AH represent parallel planes; AC and DF, those which are inclined.



Remarks. *a.* If IH be drawn parallel to CD, it will also be parallel to AB; for it is evident from the very definition above, that *lines, or planes, which are parallel to the same, are parallel to each other.*

b. As two straight lines having the same direction cannot pass through the same point (§ 6. *b*), it is evident that *parallel lines can never meet.*

V. A.



a.) Draw the straight line AB.

From A, draw the straight line AC towards B, as in

(o) Gr. παράλληλος, *along side of each other*, from παρά and ἀλλήλων.
(p) Lat. inclino, *to lean.*

No. 1. Does AC *diverge* at all from AB (that is, depart or differ from it in direction)? Is there any opening between them, or do they form one and the same straight line?

Draw AC diverging somewhat from AB, as in No. 2. Is there now any opening between them? Do they make a corner at A?

A corner is called in Geometry an ANGLE. — Is the corner, or angle, at A a *sharp* or a *blunt* one?

Draw AC diverging more from AB, as in No. 3. Is the angle sharper or blunter than in No. 2?

Draw AC diverging still more from AB, as in No. 4. Is the angle sharper or blunter than in No. 3?

As AC diverges more from AB, does the angle become sharper or blunter?

b.) If you should lengthen the lines in No. 2, 3, or 4, would it make the corners, or angles, any sharper or blunter?

If you should shorten the lines, would it make the angles any sharper or blunter?

Does the length of the lines make any difference in the sharpness or bluntness of the angles?

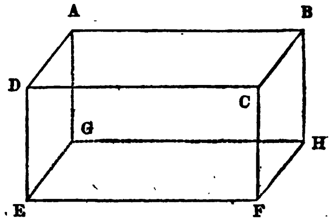
The blunter an angle is, the *greater* it is said to be, and the sharper it is, the *less*, without any regard to the length of the lines which form it. — In which of the Nos. above is there no angle? In which is the angle the greatest? In which the least?

As AC diverges more from AB, does the angle become greater or less?

Draw four angles, all equal, but with the lines of different lengths.

Draw four angles, no two equal, but all formed by lines of the same length.

c.) How many angles has the surface ABCD of the block AF?



How many has the surface CDEF?

How many angles has each surface of the block?

How many have all the surfaces together?

How many plane angles (i. e. angles made by lines in a plane) are there, then, upon the block?

At how many points upon the block are these angles made? How many are made at each point?

d.) How many surfaces diverge from the point D of the block? from the point A? from each of the angular points of the block?

Do the three surfaces diverging from D make a *solid corner* at this point? — Such a corner is called, in Geometry, a SOLID ANGLE.

How many solid angles has the block AF?

By how many planes is each of these solid angles made? How many plane angles does each contain?

How many edges, or boundary-lines of surfaces, come together at the point D? at the point A? at each angular point of the block?

e.) Do the planes AC and DF diverge from each other along the whole line DC?

From what line do the planes AC and AE diverge? AE and DF?

The divergence of two surfaces from a common line forms a DIEDRAL ANGLE. — How many diedral angles does the plane AC make with other planes? the plane DF? each plane upon the block?

How many planes are required to form each diedral angle?

How many diedral angles do all the planes upon the block form?

f.) How many plane angles has the block? How many diedral angles? How many solid angles?

How many planes are concerned in each of the plane angles? in each of the diedral angles? in each of the solid angles?

Point out plane angles upon the ceiling, sides, and floor of this room.

Point out diedral angles belonging to the room.

Point out solid angles belonging to the room.

If the two leaves of a sheet of paper are regarded as planes, what kind of an angle is made by opening them?

Bend a piece of paper so as to represent different diedral angles. Represent with paper different solid angles.

Cut an apple into various figures, and show what plane angles belong to each; what diedral angles; what solid angles. Show by how many surfaces each solid angle is formed.

V. B.

§ 8. **Difference of direction**, as it is capable of being measured, constitutes a *dimension* (§ 1. *e*) and gives rise to a new class of *magnitudes*, termed **angles**[q].

(q) Lat. angŭlus, *corner*.

REMARK. Difference of direction, from the most obvious mode of viewing and estimating it (§ 9. *c*), has received, for convenience, the name DIVERGENCE[r].

§ 9. An **ANGLE** is formed by the divergence, either of *lines from a point*, or of *surfaces from a point or line*. The lines or surfaces forming the angle are termed its SIDES; and when they diverge from a point, that point is termed the **VERTEX**[s], or **ANGULAR POINT**.

REMARKS. *a*. An angle is *greater* or *less*, according as the *difference of direction* in the lines or surfaces which form it is *greater* or *less*, without any regard to the length of the lines, or extent of the surfaces; or, in other words, *an angle is measured by the divergence of its sides.*

b. An angle formed by the divergence of two lines from a point is termed a LINEAR ANGLE; and, if it lies in a plane figure, a PLANE ANGLE. It is *rectilinear*, when both its sides are straight lines; and *curvilinear*, when one or both are curves. An angle formed by the divergence of surfaces from a point is termed a SOLID ANGLE; and one formed by the divergence of two surfaces from a line, a DIEDRAL[t] ANGLE. — For the present, our attention will be confined to plane rectilinear angles, and when the simple term *angle* is used, it must be understood in this sense.

c. Difference of direction constitutes either *divergence* or *convergence*[u], according as the direction is reckoned *from* or *towards* the angular point. In most cases, however, it is more natural to reckon it *from* this point; and therefore the word *divergence* is employed as a convenient term to express, in general, *angular dimension*, or *difference of direction*. It will be understood, that the direction is reckoned *from* the angular point, unless notice is given to the contrary.

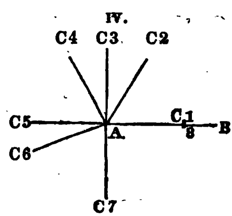
(r) Lat. divergo, *to separate*, or *go different ways from one point.* (s) L. vertex, *turning-point*, *extreme point*, *peak*, *summit.* (t) Gr. δίς, *twice*, ἕδρα, *seat*, *base.* (u) L. convergo, *to come together*, *to tend towards the same point.*

d. The difference of direction between lines or surfaces is the same, whether they actually meet at an angular point, or not. For measurement, however, it is always referred to such a point; which can readily be done, either by producing the lines or surfaces till they meet, or by drawing parallels.

e. As all the forms of extension may be regarded as originating in the motion of a point (§ 3. *d*), so all the forms of direction and divergence may be traced back to the relative direction of points. From the relative direction of the points in a line, arises the direction of the line; from the different direction of two lines, a linear angle; from the direction of the lines in a surface, the direction of the surface; from the different direction of two surfaces, a diedral angle; and from the combination of linear and diedral angles, a solid angle.

VI. A.

a.) Draw the straight line AB. Now take a wire or string, and fasten it at A, to represent a second line AC joined to AB at A (§ 5. *g*).



First, let AC lie upon AB, as in the position marked 1 upon the diagram. Does AC make any angle with AB, or do they form one and the same straight line?

Let AC revolve about the fixed point A, until it comes to the direction AC2. In revolving from 1 to 2, has AC been in every direction which is possible between AC1 and AC2? Has it then made every angle with AB which is possible between 1 and 2?

Let AC continue to revolve, until it comes to the direction AC3. In this position, does it make a greater or less angle with AB than in the position AC2.

As it revolves, does the angle which it makes with AB increase or diminish? In other words, does its divergence from AB become greater or less?

Let AC revolve to the position AC 4. Has its divergence from AB increased or diminished? Has it made with AB every angle which is possible between 1 and 4?

b.) Let AC revolve to the position AC 5, in which it forms a straight line with AB. How is the direction of AC now related to that of AB (§ 6. 2)?

Does the direction AC 5 differ more or less, than the direction AC 4, from that of AB?

Is the divergence between AC 5 and AB greater or less than that between AC 4 and AB?

Has AC now made every angle with AB which is possible upon one side of a straight line?

Has it reached the greatest divergence from AB which is possible without passing beyond the direction of a straight line?

c.) Let AC continue to revolve, until it reaches the position AC 6. If you reckon, as before, through 1, 2, 3, 4, 5, is the divergence of AC 6 from AB greater or less than that of AC 5? How is it, if you reckon the opposite way, namely, through 8, 7, 6?

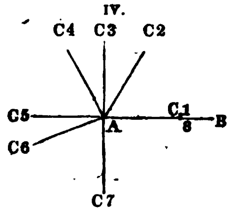
Let AC revolve to the position AC 7. How does the divergence of AC 7 from AB compare with that of AC 6, reckoned both ways?

Let AC revolve to the position AC 8; that is, let it return to AB. Has AC now made with AB every angle which is possible?

Has it passed through the whole circuit of divergence around the point A?

VI. B.

§ 10. If two straight lines AB, AC be joined at the point A, and AC first lie upon AB, and then revolve around the point A, until it returns to AB; it is evident, (1.) that this revolution will give every possible divergence of two lines, from the least to the greatest; and (2.) that AC will pass through the whole circuit of divergence around the point A. For the purposes of measurement, ***the whole divergence around a point is divided into* 360 *degrees*** [v].



NOTE. The degree (marked thus °) is divided into 60 *minutes* (′) [w]; the minute into 60 *seconds* (″) [x]; the second into 60 *thirds* (‴); &c.

QUESTIONS AND REMARKS. *a.* If the whole divergence around a point is divided into 3 equal angles, how many degrees will each contain? How many, if it is divided into 4 equal angles? into 5? into 6? into 8? into 9? into 10? into 12? into 30? into 40? into 90? into 360?

If in the figure above, the angle BAC2 is $\frac{1}{6}$ of the whole divergence around A, how many degrees does it contain? How many does BAC3 contain, if $\frac{1}{4}$ of the whole? BAC4; if $\frac{1}{3}$?

Into how many angles of 90° can the whole divergence around A be divided? Into how many of 60°? of 45°? of 40°? of 30°? of 120°? of 15°? of 36°? of 18°? of 10°? of 12°? of 5°? of 4°? of 3°? of 2°? of 1°?

(v) Fr. degré, from Lat. gradus, *step*. (w) L. minūtus, *small, minute*. (x) Fr. second, from L. secundus, *following, subordinate, next*.

§ 11. *b*. In its largest sense, the term *angle* may be applied even to the divergence between a straight line and its reverse (§ 6. 2); as, between AB and AC 5. An angle less than this is termed *concave*; greater, *convex* (see § 5. *a*).

c. As the line AC can revolve around A two ways (viz. through 1, 2, 3, 4, 5, 6, 7, 8, or, reversely, through 8, 7, 6, 5, 4, 3, 2, 1), it is evident, that its divergence from AB may be reckoned two ways (through 1, 2, 3, &c., or through 8, 7, 6, &c.); and that, consequently, it makes at each point two angles with AB, the one (except when BAC is a straight line) *concave*, and the other *convex*. Each of these angles may be termed the *reverse* of the other.

d. Point out, upon the figure above, the two angles which AC 6 makes with AB. Which is concave? Which, convex? How many degrees do they both together contain?

How many degrees must any angle and its reverse together contain?

If the number of degrees in an angle is given, how do you find the number of degrees in its reverse?

What is the reverse of an angle of 60°? of 90°? of 120°? of 180°? of 200°? of 300°?

How many convex angles can lie about a point? How many concave?

e. When the angle made by one line with another is spoken of, the *concave angle* is always meant, unless the contrary is either expressed, or obviously required by the connection.

VII.

§ 12. For the sake of greater brevity and clearness, mathematicians employ certain SIGNS[y], or SYMBOLS[z], in the place of words or phrases which often recur. Thus,

---

(y) Lat. signum. (z) Gr. σύμβολον, *token, sign*.

The Sign of	To be read,	As	For
Equality,	$=$, *equal to;*	$A=B$,	A *is equal to* B.
Superiority,	$>$, *greater than;*	$A>B$,	A *is greater than* B.
Inferiority,	$<$, *less than;*	$A<B$,	A *is less than* B.
Addition,	$+$, *plus*[a], or *and;*	$A+B$,	A *plus* B.
Subtraction,	$-$, *minus*[b], or *less;*	$A-B$,	A *minus* B.
Multiplication,	$\times$, *multiplied by*, or *into;*	$A\times B$,	A *into* B.
Division,	$\div$, *divided by;*	$A\div B$,	A *divided by* B.
Parallelism,	$\parallel$, *parallel to;*	$A \parallel B$,	A *is parallel to* B.
Perpendicularity,	$\perp$, *perpendicular to;*	$A\perp B$,	A *is perpendicular to* B.
Right Angles,	$\llcorner$, *right angle;*	$2\llcorner$,	*two right angles.*
Inference,	$\therefore$, *therefore.*		

NOTE. For the signs $^\circ$, $'$, $''$, $'''$, see § 10. N.; for $\approx$ and $\S$, see § 13; for the initials *hm, a, l, p, t,* see §§ 13, 14. *e.*

REMARKS. *a.* (1.) $A+B$ may also be read, *the sum of* A *and* B, or, A *together with* B; $A-B$, *the difference of* A *and* B, or, more strictly, *the remainder after taking* B *from* A; $A\times B$ (also written $A\,.\,B$), *the product*[c] *of* A *and* B; and $A\div B$ (for which the fractional[d] form $\frac{A}{B}$ is more common), *the quotient*[e] *of* A *by* B. (2.) The sign of multiplication is commonly omitted between numbers and letters, and between algebraic letters; thus, 3AB, for $3\times AB$, or *three times* AB; *abc*, for $a\times b\times c$. (3.) When a quantity (as A) is multiplied into itself, it is common to write A^2 for $A\times A$, A^3 for $A\times A\times A$, &c., the number above (termed the *exponent*[f]) showing how many times the quantity is taken as a factor[g]. A^2 is read, A *square*, or *second power;* A^3, A *cube*, or *third power*. (4.) Parentheses often inclose two or more quantities, or a line (called a *vincŭlum*[h]) is drawn over them, to show that they are to be taken together; thus $(A+B)\times(A-B+C)$, or $\overline{A+B}\times\overline{A-B+C}$, represents the product of $A+B$ into $A-B+C$. (5.) Two

(a) Lat., *more.* (b) Lat., *less.* (c) L. productus, *produced,* sc. by the multiplication. (d) L. fractio, from frango, *to break,* as whole numbers are *broken* to make fractions. (e) L. quoties, or quotiens, *how many times,* as it shows how many times the divisor is contained in the dividend. (f) L. expōnens, *setting forth, exposing, showing.* (g) Lat., *maker, producer,* sc. by multiplication. (h) Lat., *tie, bond.*

quantities connected by the sign of equality (as, $A = B$) form what is termed an *equation*[l]; and each of the two quantities is called a *member*[k], or *side*, of the equation.

§ 13. *b.* The plan of the following work has led to the adoption of two INTERROGATIVE SIGNS, formed from the signs of equality and parallelism by changing the *straight* to *wavy* lines, which seemed appropriate to the expression of uncertainty, or question. These signs are those of,

(1.) COMPARISON OF MAGNITUDE, $\approx$; as, $A \approx B$? for, *How does* A *compare with* B *in magnitude?* i. e. *Is it identical, equal, greater*, or *less*?

(2.) COMPARISON OF DIRECTION, $\wr\wr$; as, $A \wr\wr B$? for, *How does* A *compare with* B *in direction?* i. e. *Is it in the same straight line, parallel, perpendicular, or oblique*?

NOTE. $A \approx B$? may also be read, *How is* A *related to* B *in magnitude?* or, *What is the comparative* (or, *relative*) *magnitude of* A *and* B? $A \wr\wr B$? may also be read, *How is* A *related to* B *in position?* or, *What is the comparative direction* (or, *relative position*) *of* A *and* B?

c. The initials *hm* are used for *how many?* or *how much?* as, *hm*$\llcorner$? for *how many right angles*? *hm*°? for *how many degrees?*

§ 14. *d.* In respect to the use of CAPITAL LETTERS REFERRING TO THE DIAGRAMS, it will be understood, unless the contrary is either expressed, or obviously required by the connection,

(1.) That a *single letter* denotes a POINT; as, A, for *the point* A (i. e. near which A stands). — A single letter not unfrequently denotes the *angle* at a point; and it may be used to denote a magnitude of any kind.

(2.) That *two letters* denote a LINE; as, AB, for *the line* AB (i. e. the line lying between the points A and B). — Surfaces and solids are often named by taking two letters at opposite points not connected by a straight line, as in § 3. *c.*

(l) Lat. æquatio, from æquo, *to make equal*. (k) L. membrum, *limb*.

(3.) That *three letters* denote an ANGLE; as, ABC, for *the angle* ABC (i. e. the angle made by the lines AB and BC, the letter at the vertex being always placed in the middle). — A triangle is commonly denoted by the three letters at the angular points; and three letters are also employed to denote broken and curve lines, &c.

(4.) That *four letters* denote a QUADRILATERAL; *five*, a PENTAGON; &c.

e. To the letters referring to the diagrams, a *small initial* will be often prefixed, especially when three letters denote a triangle, or a single letter, an angle; thus, *t*ABC, for *the triangle* ABC; *a*ABC, for *the angle* ABC; *a*C, for *the angle* C; *p*C, for *the point* C; *l*AB, for *the line* AB.

§ 15. Read the following:

$a\mathrm{B} \approx a\mathrm{C}$? $\quad \mathrm{B} < \mathrm{C}. \quad \mathrm{B} = \mathrm{C}. \quad \mathrm{B} > \mathrm{C}.$

$l\mathrm{AB} \lessgtr l\mathrm{CD}$? $\quad \mathrm{AB} \perp \mathrm{CD}. \quad \mathrm{AB} \parallel \mathrm{CD}.$

$\mathrm{A} + \mathrm{B} - \mathrm{C} = \frac{\mathrm{D} \times \mathrm{E}}{\mathrm{F}}.$

$(\mathrm{A} \times \mathrm{B}) \div (\mathrm{C} - \mathrm{D}) > (\mathrm{E} + \mathrm{F}) - (\mathrm{G} - \mathrm{H} + \mathrm{I}).$

$\mathrm{A}^2 + \overline{\mathrm{A} + \mathrm{B}}^2 < \overline{\mathrm{A} - \mathrm{B} + \mathrm{C}}^2.$

$a\mathrm{DEF} = hm\llcorner$? $\therefore a\mathrm{DEF} = hm^{\circ}.$

VIII.

§ 16. The whole work of the geometer may be resolved into the comparison of magnitudes. His success, therefore, will depend upon his sagacity in finding out ways of comparing them, and his accuracy in making the comparison.

REMARKS. *a.* The comparison of magnitudes may be either more or less particular. Thus, we may simply say, in general, that two magnitudes are equal, or that one is greater, or is less, than the other. Or, of two unequal magnitudes, we may show *how much* the one is greater, or is less, than the other; or *how many times* the one contains, or

is contained in, the other. Or we may compare a magnitude with a fixed standard, and say that the length of a line is so many inches, or feet, or yards, &c.; that a surface contains so many square inches, feet, yards, &c.; that a solid contains so many cubic inches, feet, yards, &c.; or, that an angle contains so many degrees, or parts of a degree.

b. The objects of which Geometry treats may be compared in respect to *extent*, *direction*, *position*, and *form;* but this may all be resolved into the measurement of extension and direction, or, in other words, of *lines*, *surfaces*, *solids*, and *angles* (§§ 3, 8). Extent is of three kinds, according to the number of its dimensions (§ 3); *linear* (extent of line), *superficial* (extent of surface), and *solid*. The first is termed LENGTH; the second, AREA[l], OR SUPERFICIAL CONTENT[m]; and the third, SOLIDITY[n], VOLUME[o], OR SOLID CONTENT.

c. Two objects compared are said to *agree* in size, form, direction, position, &c., when they have the *same* size, form, &c. Solids, surfaces, &c., are termed *equal*[p], when they agree in *size*, without respect to form; *similar*[q], when they agree in *form*, without respect to size; *identical*[r], when they agree in *both size and form*, or are both equal and similar. In other words, identical figures *agree in all their parts;* that is, they have the same number of surfaces, lines, and angles, which are all equal each to each, and are similarly situated in respect to each other. — Some apply the term *equal* to such figures only as are equal in all their parts, terming those which are simply equal in size *equivalent*[s].

d. The distinction between equality and identity does not apply to straight lines, or to plane rectilinear angles; for it is evident, that, if these are equal, they are also identical.

e. The relation of one thing to another in respect to size is termed RATIO[t]. We may compare not only magnitudes

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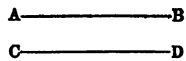
(l) Lat., *threshing-floor; any open surface.* (m) L. contentus, *contained.* (n) L. solidĭtas. (o) L. volūmen, *that which is rolled together*, from volvo, *to roll.* (p) L. æquālis. (q) L. simĭlis, *like.* (r) Fr. identique, from L. idem, *the same.* (s) L. æquus, *equal*, and valeo, *to be worth.* (t) L., *reckoning, relation.*

themselves, but also their ratios. For the present, however, we shall confine ourselves to the simple comparison of magnitudes themselves.

§ 17. There are several methods of comparing two magnitudes, as will appear from the following directions.

I. First, see if one of the magnitudes can be *applied*[u] to the other (that is, placed upon it, or put into the same place with it), for direct measurement. This method is termed APPLICATION, or SUPERPOSITION[v], and, when practicable, is the most direct method of comparison.

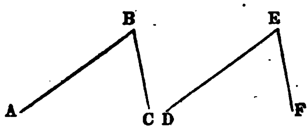
a.) Suppose the line AB to be placed upon the line CD with the point A upon the point C. If, now, *p*B falls *upon* *p*D, how do the lines compare in length? How do they compare,



if B falls *within* D (that is, nearer to C than D is)? How, if B falls *without* D (that is, farther from C than D is)?

Suppose AB to be placed upon CD with B upon D. Now, if AB be equal to CD, where will A fall? Where will it fall, if AB be greater? Where, if it be less?

b.) Suppose the angle ABC to be placed upon the angle DEF with *p*B upon *p*E, and the side BA upon ED. If, now, BC falls upon EF, how do the angles compare?

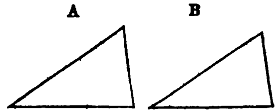


How, if BC falls within EF? How, if it falls without?

(u) Lat. applico, *to fold upon, bring to.* (v) L. super-positio, *placing upon.*

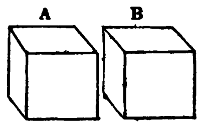
Suppose aABC to be placed upon aDEF, with B upon E, and BC upon EF. Now, if ABC ≐ DEF, where will BA fall? Where will it fall, if ABC > DEF? Where, if ABC < DEF?

c.) Suppose the surface A to be placed upon the surface B. If every line and angle of A falls upon the corresponding line and angle of B, how do the two surfaces compare? Are they more than simply equal (§ 16. *c*)?



If every part of A falls upon B, without covering the whole of B, how do they compare? How, if A covers more than the whole of B?

d.) Suppose the solid A to be put into the place of the solid B. If it precisely fills the space occupied by B, how do they compare? If every part of A falls within the space occupied by B, without filling the whole of this space, how do they compare? How, if A more than fills the space of B?

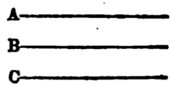


Remarks. *a.* If two magnitudes, applied to each other, agree throughout in position and extent, they are said to *coincide*. It is evident, that *magnitudes which coincide are identical* (§ 16. *c*).

b. Actual superposition can only be of limited use, and cannot possess that accuracy which Geometry requires. The superposition, therefore, which is contemplated in this science, is the work of the mind, and not of the hands (see § 5. *g*); and all the language employed in regard to it must be so understood.

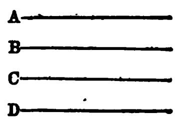
§ 18. II. Secondly, if the two magnitudes cannot be immediately compared, see if you can *compare them both with a third*; or if you can compare them with two or more magnitudes, which can themselves be compared together.

a.) Let A and C be two magnitudes (as, for example, two lines) which cannot be directly compared with each other, but which can both be compared with a third, B. Then,



If $A = B$, and $B = C$, $A \gtreqless C$?
If $A = B$, and $B > C$, $A \gtreqless C$?
If $A = B$, and $B < C$, $A \gtreqless C$?
If $A < B$, and $B = C$, $A \gtreqless C$?
If $A > B$, and $B = C$, $A \gtreqless C$?
If $A > B$, and $B > C$, $A \gtreqless C$?
If $A < B$, and $B < C$, $A \gtreqless C$?

b.) Let A and D be two magnitudes (as two lines) which can neither be directly compared with each other, nor both with a third, but can be compared with two others, B and C, which can themselves be compared. Then,



If $A = B$, $B = C$, and $C = D$; $A \gtreqless D$?
If $A = B$, $B = C$, and $C > D$; $A \gtreqless D$?
If $A = B$, $B = C$, and $C < D$; $A \gtreqless D$?
If $A = B$, $B < C$, and $C = D$; $A \gtreqless D$?
If $A = B$, $B > C$, and $C = D$; $A \gtreqless D$?
If $A > B$, $B = C$, and $C = D$; $A \gtreqless D$?
If $A < B$, $B = C$, and $C = D$; $A \gtreqless D$?
If $A = B$, $B > C$, and $C > D$; $A \gtreqless D$?

$$\text{If } A = B,\ B < C, \text{ and } C < D;\ A \gtreqless D?$$
$$\text{If } A > B,\ B = C, \text{ and } C > D;\ A \gtreqless D?$$
$$\text{If } A < B,\ B = C, \text{ and } C < D;\ A \gtreqless D?$$
$$\text{If } A > B,\ B > C, \text{ and } C = D;\ A \gtreqless D?$$
$$\text{If } A < B,\ B < C, \text{ and } C = D;\ A \gtreqless D?$$
$$\text{If } A > B,\ B > C, \text{ and } C > D;\ A \gtreqless D?$$
$$\text{If } A < B,\ B < C, \text{ and } C < D;\ A \gtreqless D?$$

REMARK. In the comparison of magnitudes, (*a*) *an equal may always take the place of its equal without affecting the result;* and (*b*) *a greater than a greater, or a less than a less, is, of course, still greater, or still less.* Hence, in a series of magnitudes successively compared as above, any one of the series may be *cancelled*, or struck out, in connection with the sign of equality (=); and any one may be cancelled in connection with a sign of inequality (> or <), if also preceded or followed by the same sign. Thus,

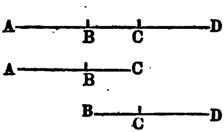
$$\begin{array}{l}
\text{If } A > B,\ B = C,\ C > D,\ D > E,\ E = F, \text{ and } F > G; \\
\text{then } A > \quad C,\ C > \quad E,\ E \quad > G; \\
\text{then} \qquad A > G.
\end{array}$$

See § 20.

§ 19. III. Thirdly, see if you can compare the two magnitudes by means of some *addition*, *subtraction*, *multiplication*, or *division*.

A. Equally performed upon both.

a.) Let AB and CD be two magnitudes (as two lines) which cannot be compared except by making the same or an equal *addition* BC to both. Then,



$$\text{If } AC = BD,\ AB \gtreqless CD?$$
$$\text{If } AC > BD,\ AB \gtreqless CD?$$
$$\text{If } AC < BD,\ AB \gtreqless CD?$$

b.) Or, on the other hand, let AC and BD be two magnitudes which cannot be compared, except by making the same or an equal *subtraction* BC from both. Then,

$$\text{If } AB = CD,\ AC \gtreqless BD?$$
$$\text{If } AB < CD,\ AC \gtreqless BD?$$
$$\text{If } AB > CD,\ AC \gtreqless BD?$$

c.) Let A and B be two magnitudes which cannot be compared, except by *multiplying* both by the same or by equal quantities; as by the numbers 2, 3, 4, &c. Then,

$$\text{If } 2A > 2B,\ A \gtreqless B?$$
$$\text{If } 3A = 3B,\ A \gtreqless B?$$
$$\text{If } 4A < 4B,\ A \gtreqless B?$$

d.) Or, on the other hand, let A and B be two magnitudes which cannot be compared, except by *dividing* both by the same or by equal quantities; as, by the numbers 2, 3, 4, &c. Then,

$$\text{If } \frac{A}{2} = \frac{B}{2},\ A \gtreqless B?$$
$$\text{If } \frac{A}{3} > \frac{B}{3},\ A \gtreqless B?$$
$$\text{If } \frac{A}{4} < \frac{B}{4},\ A \gtreqless B?$$

B. Performed upon one only.

$$\text{If } A + C = B,\ A \gtreqless B?$$
$$\text{If } A + C < B,\ A \gtreqless B?$$
$$\text{If } A = B + C,\ A \gtreqless B?$$
$$\text{If } A - C = B,\ A \gtreqless B?$$
$$\text{If } A - C > B,\ A \gtreqless B?$$
$$\text{If } A = B - C,\ A \gtreqless B?$$
$$\text{If } 2A = B,\ A \gtreqless B?$$
$$\text{If } 3A < B,\ A \gtreqless B?$$
$$\text{If } A = 4B,\ A \gtreqless B?$$
$$\text{If } \frac{A}{2} = B,\ A \gtreqless B?$$

If $\frac{A}{3} > B$, A ≷ B?

If $A = \frac{B}{4}$, A ≷ B?

C. Performed upon both, but unequally.

If $A + B = C + D$, and $B > D$; A ≷ C?

If $A + B = C + D$, and $B < D$; A ≷ C?

If $A + B > C + D$, and $B < D$; A ≷ C?

If $A + B < C + D$, and $B > D$; A ≷ C?

If $A - B = C - D$, and $B > D$; A ≷ C?

If $A - B = C - D$, and $B < D$; A ≷ C?

If $A - B > C - D$, and $B > D$; A ≷ C?

If $A - B < C - D$, and $B < D$; A ≷ C?

If $A \times B = C \times D$, and $B > D$; A ≷ C?

If $A \times B = C \times D$, and $B < D$; A ≷ C?

If $A \times B > C \times D$, and $B < D$; A ≷ C?

If $A \times B < C \times D$, and $B > D$; A ≷ C?

If $\frac{A}{B} = \frac{C}{D}$, and $B > D$; A ≷ C?

If $\frac{A}{B} = \frac{C}{D}$, and $B < D$; A ≷ C?

If $\frac{A}{B} > \frac{C}{D}$, and $B > D$; A ≷ C?

If $\frac{A}{B} < \frac{C}{D}$, and $B < D$; A ≷ C?

REMARKS. 1. In regard to these operations performed (equally or unequally) upon equal or unequal magnitudes, it is evident, that EQUAL OPERATIONS *affect*, and EQUAL MAGNITUDES *are affected*, *equally*; UNEQUAL, *unequally*. Hence, if *equal operations* are performed upon *equal magnitudes*, the *results are equal*; and, if *equal operations* are performed upon *unequal magnitudes*, or if *unequal operations* are performed upon *equal magnitudes*, the *results are unequal*. When unequal operations are performed upon unequal magnitudes, no general rule can be given in regard to the results. See § 20.

§ 20. 2. The axiom upon which Methods II. and III. (§§ 18, 19) are founded may be thus stated in general terms:

EQUALS SUSTAIN LIKE RELATIONS, AND RECEIVE AND PRODUCE LIKE EFFECTS; AND UNEQUALS, UNLIKE. Hence,

(*a*) Things which are *equal to the same* are equal to each other.

(*b*) An *equal to a greater*, is greater; *to a less*, less. A *greater than an equal* is greater; a *less*, less.

(*c*) A *greater than a greater* is still greater; a *less than a less*, still less.

(*d*) If *equals are added to equals*, the sums are *equal*.

(*e*) If *equals are added to unequals*, the sums are *unequal*.

(*f*) If *equals are subtracted from equals*, the remainders are *equal*.

(*g*) If *equals are subtracted from unequals*, or *unequals from equals*, the remainders are *unequal*.

(*h*) If *equals are multiplied by equals*, the products are *equal*.

(*i*) If *equals are multiplied by unequals*, or *unequals by equals*, the products are *unequal*.

(*k*) If *equals are divided by equals*, the quotients are *equal*.

(*l*) If *equals are divided by unequals*, or *unequals by equals*, the quotients are *unequal*.

§ 21. 3. It is also evident, that,

(*a*) If *the same quantity be both added and subtracted*, there is no change of value. Thus, $A + B - B = A$; and if $B = C$, then $A + B - C = A$ (§ 18. R.).

(*b*) If a quantity be *both multiplied and divided by the same*, its value is not changed. Thus, $A \times B \div B = A$, and, if $B = C$, then $A \times B \div C = A$.

(*c*) The WHOLE is *greater than any of its parts*, and is *equal to the sum of its parts*.

§ 22. 4. The axioms in §§ 20, 21 belong to Arithmetic and Algebra no less than to Geometry. One of their simplest applications is to the finding of the value of two quantities in

terms of their sum and difference (that is, in expressions in which the sum and difference are employed). Let g represent the greater of the two quantities; l, the less; s, their sum; and d, their difference. Then,

$$g + l = s,$$
$$g - l = d.$$

By adding the equals in the second equation to the equals in the first[w] (§ 20. d), we obtain

$$2g = s + d;$$

and by dividing each member of this equation by 2 (§ 20. k),

$$g = \tfrac{1}{2}s + \tfrac{1}{2}d;$$

that is, *the greater of two quantities is equal to half their sum plus half their difference.*

By subtracting the second equation from the first (§ 20. f), we obtain

$$2l = s - d;$$

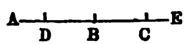
and by dividing this equation by 2,

$$l = \tfrac{1}{2}s - \tfrac{1}{2}d;$$

that is, *the less of two quantities is equal to half their sum minus half their difference.*

For illustration, take two lines, AB and BC, and join them at B, making AC their sum. From AB take BD = BC, leaving AD their difference. What line is the sum minus the difference ($s - d$)? As BD = BC, what part of DC is BC?

Extend AC to E, making CE = AD. What line is the sum plus the difference ($s + d$)? As AD = CE, and DB = BC, AB ≈ BE? Then, what part of AE is AB?



(w) This is commonly termed *adding the equations.*

§ 23. IV. Fourthly, see if any supposition can be made, every supposition contrary to which may be shown to involve an *absurdity*, or *impossibility*.

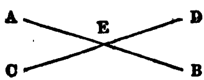
REMARK. It is evident, that (*a*) *any supposition which involves an absurdity must be false;* and (*b*) *any, to which every contrary supposition involves an absurdity must be true.* This method of proof is termed REDUCTIO AD ABSURDUM[x], and is often useful as a substitute for direct proof, or as collateral with it.

(x) Lat., *reducing to an absurdity.*

PART SECOND.

§ 24. **DEFINITIONS.** Angles are termed ADJACENT[a], when they *lie side by side* at the same point; VERTICAL[b], when they are made by the intersection of two lines, and lie *opposite* to each other.

Thus, AEC and AED are *adjacent angles*, of which AE is the *common*, or *inner*, *side*, and CE and ED, the *outer sides*. AEC and DEB, or AED and CEB, are *vertical angles*.

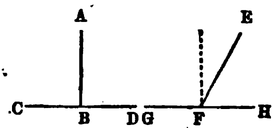


How many pairs of adjacent angles can you find in the figure? Name them. How many pairs of vertical angles can you find? Draw other angles upon the board, and show which are adjacent, and which vertical.

§ 25. If one straight line meeting another makes the two adjacent angles *equal*, these angles are termed RIGHT ANGLES[c]; and the lines are said to be PERPENDICULAR[d] to each other. If it makes the adjacent angles *unequal*, both the angles and the lines are termed OBLIQUE[e]. An oblique angle *less* than a right angle is termed ACUTE[f]; and one *greater*, OBTUSE[g].

(a) Lat. adjaceo, *to lie contiguous*, or *close by*. (b) So called from their being simply joined at the *vertex*. (c) So called, because one line stands *upright* upon the other (regarded as lying horizontally). (d) L. perpendicu-lāris, *plumb*, from perpendicŭlum, *plumb-line*. (e) L. obliquus, *slanting*. (f) L. acūtus, *sharp*. (g) L. obtūsus, *blunt*.

ABC and ABD represent *right angles* made upon CD by the *perpendicular* AB. EFG and EFH are *oblique angles*, the first *obtuse* and the second *acute*, formed upon GH by the *oblique line* EF.



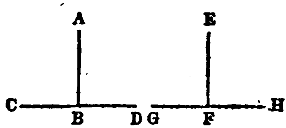
Draw various angles, and show which are right, and which are oblique; which are acute, and which obtuse. Show which of the lines employed are perpendicular to each other, and which are oblique.

Proposition I.

§ 26. *Given*, any right angles ABC, ABD, EFG, EFH.

Required, ABC ≋ ABD ≋ EFG ≋ EFH (that is, the comparative magnitude of the angles ABC, ABD, EFG, EFH; see § 13).

V.



Place the first figure upon the second, with the *point* B upon the *point* F, and the *line* BC upon the *line* FG. As CD and GH are straight lines, where must BD fall? 6. *c.*[h]

Does CD now form one and the same straight line with GH? As *p*B[i] lies upon *p*F, do AB and EF both meet this line at the same point, making the adjacent angles equal? Where then must AB fall?

With what angle must ABC coincide; and with what, ABD?

∴ ABC ≋ EFG, and ABD ≋ EFH? 17. *a.*

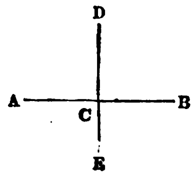
(h) Numbers and letters thus placed at the end of a line refer to sections and their parts. (i) See § 14. *e.*

But, as ABC and ABD are right angles, ABC ≋ ABD? 25.
∴ ABC ≋ ABD ≋ EFG ≋ EFH? 20. *a.*

§ 27. Theorem I. *All right angles are equal.*

[Proved by superposition.]

VI.



§ 28. *a.*) From any point C in any straight line AB, draw CD above, and CE below, each perpendicular to AB (§ 25). Then, DCA ≋ DCB ≋ ECA ≋ ECB?

Into how many equal angles is the divergence around C divided?

How many degrees does each of these angles contain (§ 10. *a*)?

How many degrees must every right angle contain? 27.

b.) The divergence between CB and its reverse CA (reckoned either way, § 11. *c*) = *hm*∟[k]? = *hm*°?

If this divergence (on either side of AB) be divided into 3 equal angles, how many degrees will each contain? How many, if it be divided into 4 equal angles? into 5? into 6? into 9? into 10? into 18? into 45? into 60?

Into how many angles of 90° each can the divergence between CB and its reverse CA be divided? Into how many of 60°? of 45°? of 30°? of 36°? of 20°? of 10°?

The sum of any angles made at the point C on one side of AB = *hm*∟? = *hm*°?

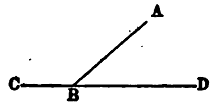
c.) The whole sum of any number of angles made *around* the point C = *hm*∟?

How many obtuse angles can lie about a point? How many acute?

§ 29. *d.*) Corollaries. (1.) *A right angle contains* 90°. (2.) *The angles made upon one side of a straight line at any point are together equal to two right angles, or contain* 180°. (3.) *All the angles around any point are together equal to four right angles.*

(k) See §§ 12, 13. *c.*

§ 30. *e.*) DEFINITION. When one straight line meets another, each of the adjacent angles is termed the SUPPLEMENT[1] of the other. Thus, ABC is the supplement of ABD; and, conversely, ABD is the supplement of ABC.



f.) An angle and its supplement = *hm* ∟ ? = *hm* ° ? 29. 2.

If the number of degrees in an angle is given, how do you obtain the number of degrees in its supplement?

What is the supplement of an angle of 60° ? of 90° ? of 110° ? of 45° ? of 20° ? of 170° ?

If an angle is a right angle, what kind of an angle must its supplement be? If acute? If obtuse?

g.) In general, any angle is called the *supplement* of another, however they may be situated, if the two are together equal to two right angles, or contain 180°; and angles so related are termed *supplementary*.

h.) If two angles are equal, how do their supplements compare?

If two angles are unequal, how does the supplement of the greater compare with the supplement of the less?

If the difference of two angles is 10°, what is the difference of their supplements?

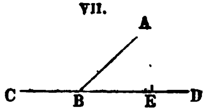
Is the supplement of an angle of 25° greater or less than the supplement of an angle of 50°, and how much?

Equal angles have equal supplements; greater, less; and less, greater.

PROPOSITION II.

§ 31. *Given*, two adjacent angles, ABC, ABD, together equal to two right angles.

VII.



Required, CB ⌇ BD (that

(1) Lat. supplementum, from suppleo, *to fill up, supply*.

is, the relative position of CB and BD; see § 13).

Produce[m] CB to E. Then, ABC + ABE = *hm*∟? 29. 2.

∴ ABC + ABE ≋ ABC + ABD? 20. *a*.

∴ ABE ≋ ABD? 20. *f*.

Upon what line then must BE fall?

∴ As CB is in the same straight line with BE, CB ‖ BD?

§ 32. THEOR. II. *If two adjacent angles are together equal to two right angles, the outer sides form a straight line.*

[The converse[n] of § 29. 2.]

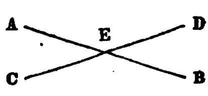
a.) If two adjacent angles are both right angles, do the outer sides form a straight line? Do they form a straight line, if the two angles are together greater, or together less, than two right angles? If one angle is 60°, and the other 120°? If the two angles are 50° and 130°? 90° and 70°? 90° and 100°? 60° and 80°? 75° and 150°?

b.) If the angle ABC contains 120°, how many degrees must ABD contain, in order that CB and BD may form a straight line? How many, if ABC contains 110°? 125°? 130°?

PROPOSITION III.

§ 33. *Given*, two straight lines, AB and CD, intersecting at E.

VIII.



Required, *a*AEC ≋ *a*DEB, and *a*AED ≋ *a*CEB.

(m) See § 5. *c*. (n) Lat. conversus, *turned about*. Of two propositions or sentences, each is said to be the *converse* of the other, when the *condition* of the first is the *conclusion* of the second, and the *conclusion* of the first is the *condition* of the second; or when, in like manner, subject and predicate change places.

	AEC + AED = *hm* ∟?	29. 2.
And	DEB + AED = *hm* ∟?	
∴	AEC + AED ≈ DEB + AED?	20. *a*.
∴	AEC ≈ DEB?	20. *f*.

Show the same, taking a different angle in the place of AED. Show, in like manner, AED ≈ CEB.

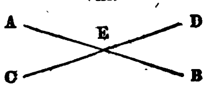
§ 34. Theor. III. *Vertical angles are equal.*

[Because each of two vertical angles makes, with one of the intermediate angles, the same sum, viz. 180°.]

a.) If AEC contains 50°, *hm*° does each of the other angles in the figure contain? If it contains 60°? 75°? 45°? 90°?

If one of the angles made by two intersecting lines is a right angle, what must they all be? If one is acute, what must the rest be? If one is obtuse?

VIII.



§ 35. *b*.) *Given*, at the point E four angles, of which those that are opposite are equal; i. e. AEC = DEB, and AED = CEB.

Required, CE ⫶ ED, and AE ⫶ EB.

	AEC + AED ≈ DEB + CEB?	20. *d*.
But	AEC + AED + DEB + CEB = *hm* ∟?	29. 3.
∴	AEC + AED = *hm* ∟?	
∴	CE ⫶ ED?	32.
And	AED + DEB = *hm* ∟? ∴ AE ⫶ EB?	

If, of four angles at the same point, those which are opposite are equal, they are formed by the intersection of two straight lines.

---

§ 36. Definitions. A *plane figure inclosed by straight lines* is termed a POLYGON° (or simply, a RECTILINEAL FIGURE). The lines which in-

(o) Gr. πολύγωνος, *many-angled*, from πολύς, *many*, and γωνία, *corner, angle*.

close it are termed its **SIDES** ; and the sum of these lines, its **PERIMETER**[p].

§ 37. A polygon of *three* sides is termed a **TRIANGLE**[q] ; of *four*, a **QUADRILATERAL**[r] ; of *five*, a **PENTAGON**[s] ; of *six*, a **HEXAGON**[t] ; of *seven*, a **HEPTAGON**[u] ; of *eight*, an **OCTAGON**[v] ; of *ten*, a **DECAGON**[w] ; &c.

§ 38. The sides and angles of a polygon are termed its **PARTS** ; and those which lie next to each other (taking sides and angles alternately) are termed **ADJACENT PARTS.**

Thus, in Fig. IX. (§ 39), the side AB is adjacent to the angles A and B, and the angle A is adjacent to the sides AB and AC.

What are the parts adjacent to the side BC? to AC? to the angle B? C? to the side EF? to the angle D?

How many parts has a triangle? a quadrilateral? a hexagon? a decagon?

In every polygon, how does the number of the sides compare with the number of the angles?

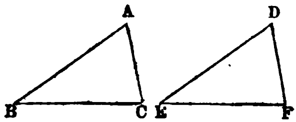
PROPOSITION IV.

§ 39. *Given*, two triangles, ABC, DEF, agreeing in three adjacent parts (i. e. having three adjacent parts of the one equal to three adjacent

(p) Gr. περίμετρος, *measure round*, from περί, *around*, and μέτρον, *measure*. (q) Lat. triangŭlus, *three-cornered*, from tres, *three*, and angŭlus. (r) L. quadrilatĕrus, *four-sided*, from quatuor, *four*, and latus, *side*. (s) Gr. πεντάγωνος, *five-cornered*, from πέντε, *five*, and γωνία. (t) Gr. ἕξ, *six*. (u) Gr. ἑπτά, *seven*. (v) Gr. ὀκτώ, *eight*. (w) Gr. δέκα, *ten*.

parts of the other, each to each; see § 16. *c*).

IX.



Required, *t*ABC ≈ *t*DEF.

I. Let the three parts be two sides and the included angle; e. g. AB = DE, AC = DF, and *a*A = *a*D.

Place *t*ABC upon *t*DEF, with *p*A upon *p*D, and the side AB upon DE.

Then, as AB = DE, where will *p*B fall?

As *a*A = *a*D, where will AC fall?

As AC = DF, where will *p*C fall?

Where, then, will BC fall? 6. *c*.

Do the two triangles coincide throughout?

How then do they compare in every part? 17. *a*, 16. *c*.

What side is equal to BC? What angle to *a*B? to *a*C?

II. Let the three parts be a side and the two adjacent angles; e. g. BC = EF, *a*B = *a*E, and *a*C = *a*F.

Place *t*ABC upon *t*DEF, with B upon E, and BC upon EF.

Then, as BC = EF, where will C fall?

As *a*B = *a*E, where will BA fall?

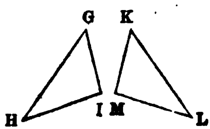
As *a*C = *a*F, where will CA fall?

Where, then, must *p*A fall?

Are the two triangles identical?

What side is equal to AB? to AC? What angle is equal to *a*A?

X.



If the triangles GHI, KLM, have GH = KL, GI = KM, and *a*G = *a*K (or HI = LM, *a*H = *a*L, and *a*I = *a*M), how would you place *t*GHI upon *t*KLM, to show that they are identical?

§ 40. THEOR. IV. *Triangles agreeing in three adjacent parts are identical.*

[Proved by superposition.]

In identical triangles, how do those angles compare which are opposite to equal sides? How do those sides compare which are opposite to equal angles?

In the identical triangles GHI and KLM (Fig. X.), mention each pair of equal parts.

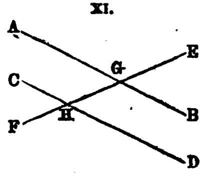
REMARK. *In identical triangles, equal angles are opposite to equal sides.*

---

§ 41. DEFINITIONS. When parallels are crossed by a single straight line, or by other parallels, the angles which are made are termed, with respect to each other, LIKE, or UNLIKE. They are termed LIKE, if they are turned either the same way or directly opposite ways; but otherwise, UNLIKE.

Thus, AGE, CHG, BGH, and DHF, form one class of like angles about the parallels AB and CD, being all turned either up or down; and EGB, GHD, AGH, and CHF, form a second class of angles which are like to each other, but unlike the first class, being all turned either to the right hand or to the left.

XI.



§ 42. Different pairs of angles about parallels have also received special names according to their situation. Thus,

a.) LIKE ANGLES. Two angles turned the same way, the one without and the other within the parallels, are termed *exterior*[x]*-interior*[y] angles; as EGB and GHD, or AGE and CHG. Two angles turned opposite ways, and both within

(x) Lat., *outer*. (y) Lat., *inner*.

the parallels, are termed *alternate*[z]*-interior* (or often simply *alternate*) angles; as, AGH and GHD, or BGH and GHC. Two angles turned opposite ways, and both without the parallels, are termed *alternate-exterior* angles; as, EGB and CHF, or AGE and DHF.

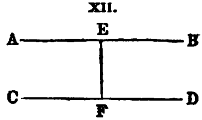
b.) Of Unlike Angles, we distinguish the *interior angles upon the same side*, as AGH and GHC, or BGH and GHD; and the *exterior angles upon the same side*, as AGE and CHF, or EGB and DHF.

Note. The above may be termed different *orders*, or *species*, of like and unlike angles.

Draw two parallels crossing two other parallels, as in Fig. XIV. (§ 45); and mention each pair of exterior-interior angles, of alternate-interior angles, of alternate-exterior angles, of interior angles upon the same side, and of exterior angles upon the same side. Mention all the angles of each of the two classes of like angles (§ 41).

Proposition V.

XII.



§ 43. I. *Given*, AB $\parallel$ CD, and EF $\perp$ AB.

Required, EF $\perp$ CD.

Let the figure AEFC be folded over upon the figure BEFD (or, in other words, be placed upon it with EF of the one figure upon EF of the other).

As aAEF = aBEF (§ 25), where will EA fall?

Does EA now form the same line with EB?

As FC has the same direction with EA, and FD with EB, have now FC and FD the same direction with each other?

Where, then, must FC fall?

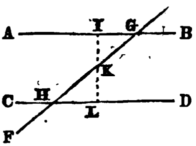
(z) Lat. alternatus, from alterno, *to vary, interchange.*

If not, how many straight lines having the same direction would pass through pF? 6. *b*.

∴ aEFC ≋ aEFD, and EF ∦ CD? 25.

§ 44. II. *Given*, AB ∥ CD, and EF, crossing them at G and H.

XIII.



Required,

AGH ≋ GHD ≋ EGB ≋ CHF;
BGH ≋ GHC ≋ AGE ≋ DHF;
and AGH + GHC, or BGH + GHD, = *hm* ∟.

(1.) If EF ⊥ AB, then EF ∦ CD? 43.

Then what are all the angles which EF makes with AB and CD? 34. *a*.

(2.) If EF is not ⊥ AB, draw IL ⊥ AB through a point K in GH, so taken that IK shall be = KL.

Then, IL ∦ CD? 43.

∴ aKIG ≋ aKLH? 27.

And aGKI ≋ aHKL? 34.

Therefore, as IK is made = KL, in what three adjacent parts do the triangles KIG and KLH agree?

∴ tKIG ≋ tKLH? 40.

∴ aIGK (or AGH) ≋ aKHL (or GHD)?

But EGB ≋ AGH, and CHF ≋ GHD? 34.

How then do these four angles compare with each other? 20. *a*.

What are the supplements of these angles? 30. *e*.

∴ BGH ≋ GHC ≋ AGE ≋ DHF? 30. *h*.

How do the four angles at G severally compare with the corresponding angles at H?

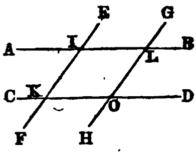
Again, as GHC = BGH,

AGH + GHC ≋ AGH + BGH? 20. *d*.

∴ AGH + GHC = *hm*∟? 29. 2.

Show the same in respect to BGH + GHD; and in respect to AGE + CHF.

XIV.



§ 45. III. *Given*, the parallels AB and CD crossing the parallels EF and GH at I, K, L, and O.

Required, a comparison of the angles at I, K, L, and O.

As AB ∥ CD, and EF crosses them, how do the angles at I severally compare with the corresponding angles at K? 44.

As AB ∥ CD, and GH crosses them, how do the angles at L severally compare with those at O?

As EF ∥ GH, and AB crosses them, how do the angles at I severally compare with those at L?

How, then, do the corresponding angles at I, K, L, and O, severally compare with each other?

What angles in the figure are each = AIK?

What angles are each = IKC?

Any angle of the one class added to any angle of the other = *hm* ∟?

How, then, is any angle of the one class related to any angle of the other class? 30. *g*.

If AIK contains 60°, how many degrees does each of the other angles contain?

§ 46. THEOR. V. *Like angles about parallels are equal; and unlike, supplementary.*

[Proved by means of superposition, identical triangles, &c.]

To this general proposition may be referred the following particulars:

(*a*) A perpendicular to one of two parallels is perpendicular to the other also.

(*b*) Two parallels make, with a third straight line, the *exterior-interior*, or (*c*) the *alternate-interior*, or (*d*) the *alternate-exterior angles* equal; and (*e*) the *interior angles upon the same side*, or (*f*) the *exterior upon the same side*, equal to two right angles.

Mention, in Fig. XIV. above, each pair of exterior-interior angles; of alternate-interior angles; of alternate-exterior angles; of interior angles upon the same side; and of exterior angles upon the same side.

(*g*) When parallels are crossed by a single straight line, or by other parallels, either all the angles are right angles; or we have one class of equal acute angles, and a second class of equal obtuse angles, which are supplements of the acute.

Draw three parallels crossing obliquely three other parallels; and point out the angles of each class.

§ 47. Remark. The distinction of *like* and *unlike angles* applies also to those angles which are formed by the simple intersection of two straight lines (the vertical angles being like, and the adjacent unlike); so that Theorems III. and V. might be united; thus:

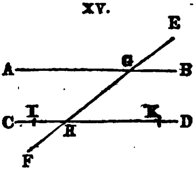
If one straight line or set of parallels crosses another straight line or set of parallels, the LIKE *angles which are formed are equal, and the* UNLIKE *supplementary.*

Proposition VI.

§ 48. *Given*, *a*AGH = *a*GHD.

Required, AB ∦ CD.

XV.



Through *p*H, draw IK ∥ AB. Are AGH and GHK like or unlike angles? Why? 41.

Of which species? 42.

∴ AGH ≈ GHK? 46. *c*.

∴ GHK ≈ GHD? 20. *a*.

Where, then, must IK fall?

∴ As AB ∥ IK, AB ∦ CD?

Show the same, there being given BGH = GHC, or EGB = GHD, or AGH + GHC = 2∟.

What other conditions being given, could you show the same?

§ 49. THEOR. VI. ***Two straight lines making, with a third, the like angles equal, or the unlike supplementary, are parallel.***

[The converse of Theorem V.]

Hence two straight lines are parallel, (*a*) when they are both perpendicular to a third; (*b*) when they make, with a third, the *exterior-interior*, (*c*) the *alternate-interior*, or (*d*) the *alternate-exterior angles* equal; or (*e*) when they make, with a third, the *interior angles upon the same side*, or (*f*) the *exterior upon the same side*, equal to two right angles.

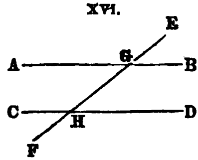
§ 50. Can two distinct straight lines, which are both perpendicular to a third, pass through the same point? *49. a, 7. b.*

COR. *Only one perpendicular to a given line can pass through a given point.*

§ 51. REMARKS. I. The equality, both of vertical angles, and of like angles formed by parallels, rests upon a common foundation; viz. that these angles arise from the difference either of the same directions or of reverse directions (§ 6. 2). The relations of these angles is well illustrated by employing subtraction to obtain their measure.

a.) The first step must of course be, to adopt some standard, by divergence from which we can estimate the directions of the several lines. Let this standard be the direction from G to B, and let the divergence from this be reckoned round the same way as in § 10, that is, from B, through E, A, and H, back again to B. Let the divergence of GE from GB be represented by e°. Then the directions

XVI.



of the several lines proceeding from pG will be represented as follows.

1st Dir. GB (as not departing from the standard) $= 0^\circ$.
Dir. GE $= e^\circ$.
Dir. GA (as the reverse of GB) $= 0^\circ + 180^\circ = 180^\circ$.
Dir. GH (as the reverse of GE) $= e^\circ + 180^\circ$.
2d Dir. GB (after an entire revolution) $= 360^\circ$.

As CD has the same direction with AB, and EF is a straight line, the directions of the lines proceeding from pH will be represented in like manner; thus, 1st Dir. HD $= 0^\circ$, Dir. HG $= e^\circ$, Dir. HC $= 180^\circ$, Dir. HF $= e^\circ + 180^\circ$, 2d Dir. HD $= 360^\circ$.

b.) As the magnitude of an angle depends simply upon the difference between the directions of its sides (§ 9. *a*), the values of the several angles around G and H may now be obtained by subtraction. Thus,

$$\left.\begin{array}{l}\text{EGB} (= \text{Dir. GE} - \text{1st Dir. GB}), \text{ or} \\ \text{GHD} (= \text{Dir. HG} - \text{1st Dir. HD}),\end{array}\right\} = e^\circ - 0^\circ = e^\circ.$$

$$\left.\begin{array}{l}\text{AGH} (= \text{Dir. GH} - \text{Dir. GA}), \text{ or} \\ \text{CHF} (= \text{Dir. HF} - \text{Dir. HC}),\end{array}\right\} = e^\circ + 180^\circ - 180^\circ = e^\circ.$$

$$\left.\begin{array}{l}\text{AGE} (= \text{Dir. GA} - \text{Dir. GE}), \text{ or} \\ \text{GHC} (= \text{Dir. HC} - \text{Dir. HG}),\end{array}\right\} = 180^\circ - e^\circ.$$

$$\left.\begin{array}{l}\text{BGH} (= \text{2d Dir. GB} - \text{Dir. GH}), \text{ or} \\ \text{DHF} (= \text{2d Dir. HD} - \text{Dir. HF}),\end{array}\right\} \begin{array}{l}= 360^\circ - (e^\circ + 180^\circ) \\ \qquad = 180^\circ - e^\circ.\end{array}$$

It will be observed, that the four angles, EGB, GHD, AGH, and CHF, have the same value, viz. e°; and also, that the four, AGE, GHC, BGH, and DHF, have the same value, viz. $180^\circ - e^\circ$, or, in other words, the supplement of e° (§ 30. *g*).

c.) Substitute for e a particular number, as 60, and the directions of the several lines will be thus represented: 1st Dir. GB or HD $= 0^\circ$, Dir. GE or HG $= 60^\circ$, Dir. GA or HC $= 180^\circ$, Dir. GH or HF $= 60^\circ + 180^\circ = 240^\circ$, 2d Dir. GB or HD $= 360^\circ$. The values of the angles will then be as follows:

EGB, or GHD, = 60° — 0°, = 60°.	Class I.	
AGH, or CHF, = 240° — 180°, = 60°.	Acute.	
AGE, or GHC, = 180° — 60°, = 120°.	Class II.	
BGH, or DHF, = 360° — 240°, = 120°.	Obtuse.	

What will be the direction of each line, and the value of each angle, if *e* be 45? if *e* be 90?

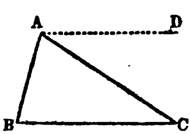
d.) A subtraction, like that above, is actually employed in finding the angle between two lines with a theodolite or compass. First, by taking sight, we find the direction of each line, as measured in degrees upon the instrument from a zero point, or from a north and south line; and then, we subtract one direction from the other to obtain the angle between them.

If in surveying we find the direction of one side of a field to be 25° upon the theodolite, and that of the adjoining side to be 85°, what is the angle between them? What is the angle, if the two directions are 45° and 160°? 60° and 179°?

§ 52. II. Without having recourse to more formal demonstration, it is evident that, if two straight lines have the same direction, they must differ equally in direction from a third straight line, and consequently must make equal angles with it; and conversely, that, if two straight lines differ alike in direction from a third, they must have the same direction with each other, or, in other words, must be parallel. — In applying this summary demonstration, the distinction in § 6. 2 must be regarded.

Proposition VII.

XVII.



§ 53. *Given*, any triangle ABC.

Required, the sum of its angles.

From A, one of the angular points of the triangle, draw AD parallel to BC, the opposite side.

What lines make angles with these parallels?

What *like angles* are made? Of which species? 42.

What *unlike angles* are made? Of which species?

Then, aBCA ≋ aCAD? 46. *c.*

∴ BCA + CAB + ABC ≋ CAD + CAB + ABC? 20. *d.*

But, CAD + CAB + ABC (or DAB + ABC) = *hm*∟? 46. *e.*

∴ BCA + CAB + ABC = *hm*∟? 20. *a.*

Show the same, drawing the parallel from A to the left hand, instead of the right.

Show the same, drawing the parallel from B; from C.

How many different ways can you draw the parallel, and show the same?

§ 54. THEOR. VII. *The three angles of any triangle are together equal to two right angles.*

[Proved by drawing from one of the points a parallel to the opposite side, and applying Theor. V.]

a.) How many degrees do the three angles of any triangle together contain? 29. I.

If two angles of a triangle are 60° and 80°, what is the third angle? What is it, if the other two are 100° and 20°? 90° and 45°? 50° and 25°?

If one angle of a triangle is 40°, and the other two are equal, *hm*° does each of them contain?

If the three angles of a triangle are equal, *hm*° does each contain?

If, in the triangle OPQ, aP is twice as great as aO, and aQ three times as great as aP, *hm*° does each contain?

If two angles of a triangle are given, how do you find the third?

b.) How is each angle of a triangle related to the sum of the other two? 30. *g.*

c.) If two angles of one triangle are equal to two angles

of another, how does the third angle of the one compare with the third angle of the other?

d.) How many right angles can a triangle have? How many obtuse? How many acute?

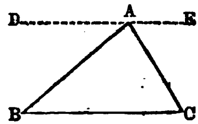
If one of the angles of a triangle is a right or an obtuse angle, what must each of the other two be? To what kind of an angle, in each case, are the other two together equal?

If one of the angles of a triangle is equal to the sum of the other two, what must that angle be?

If two angles of a triangle are equal, what kind of an angle must each be?

e.) Through A, draw DE ∥ BC. Show that the three angles of *t*ABC are equal to the three angles at *p*A, and, therefore, = 2∟.

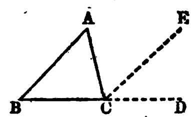
XVIII.



f.) Produce BC to D, and draw CE ∥ BA.

Show that the three angles of *t*ABC are = the three angles at *p*C, and, therefore, = 2∟.

XIX.



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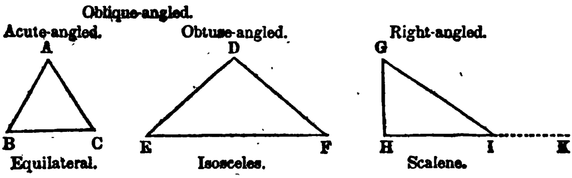
§ 55. Definitions. I. A polygon is termed *equilateral*[a], when its sides are all equal; and *equiangular*, when its angles are all equal. A triangle is termed *isosceles*[b], when it has two equal sides; and *scalene*[c], when no two of its sides are equal. It is also termed *right-*, or *obtuse-angled*, when it has one right, or one obtuse angle (§ 54. *d*); and *acute-angled*, when all its angles are acute.

(a) Lat. æquilaterālis, from æquus, *equal*, and latus, *side*. (b) Gr. ἰσοσκελής, from ἴσος, *equal*, and σκέλος, *leg*. (c) Gr. σκαληνός, *limping*, like a man whose legs are unequal.

NOTE. (*a*) Polygons which are both equilateral and equiangular are termed *regular*. (*b*) Polygons whose angles are all oblique are termed *oblique-angled*. (*c*) The shorter forms, *right triangle*, *oblique triangle*, &c., are sometimes used for *right-angled triangle*, *oblique-angled triangle*, &c.



Draw triangles of each of the different classes above mentioned.

§ 56. II. The side upon which a triangle is regarded as resting is termed its BASE[d], and the opposite angular point, its VERTEX or SUMMIT.

NOTE. (*a*) In distinction from the base, the other two sides are termed the *legs*, or sometimes simply the *two sides*. (*b*) In an *isosceles* triangle, the term *base* is specially applied to the side lying between the two equal sides.

In the triangles above, lying as they do, name the base, vertex, and legs of each.

§ 57. III. In a triangle, the side which is opposite to an angle is said to *subtend*[e] the angle.

In the triangles above, name the side subtending the angle A; E; I; C; D; H. Name the angle subtended by AC; DE; HI; BC; DF; GH.

§ 58. IV. In a right-angled triangle, the side subtending the right angle is termed the HYPOTENUSE[f]; and of the other two sides, one is often termed the BASE (§ 56), and the other the PERPENDICULAR.

In the right-angled triangle GHI, which side is the hypot-

(d) Gr. βάσις, *foundation*. (e) Lat. subtendo, *to stretch beneath*. (f) Gr. ὑποτείνουσα, *subtending*. The term *legs* is sometimes applied to the other two sides of a right triangle.

cause? As the triangle lies, which side is the base, and which the perpendicular?

§ 59. V. The angle between a side of a polygon and an adjoining side produced is termed an EXTERIOR ANGLE of the polygon; as GIK, in the figure above.

a.) In taking the sum of the exterior angles of a polygon, only one side is produced from each angular point of the polygon, as in Fig. XXI. (§ 62).

If, at any point, both sides were produced, how many exterior angles would be made? How would these compare with each other?

b.) In distinction from the exterior angles, the angles within the polygon (commonly called simply *the angles of the polygon*) are termed *interior*.

c.) aGIK + aGIH = hm∟? = hm°? 29. 2.

How, then, are these angles related to each other? 30.

REMARK. *An exterior angle is the supplement of its adjacent interior.*

In an acute-angled triangle, what kind of an angle is each exterior angle? In an obtuse-angled triangle? In a right-angled triangle?

§ 60. VI. A straight line connecting opposite angles of a polygon is termed a DIAGONAL[g].

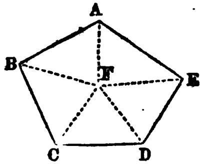
Name the diagonals in Fig. XXIII. (§ 65).

PROPOSITION VIII.

§ 61. I. *Given*, any polygon ABCDE.

Required, the sum of its interior angles.

XX.



(g) Lat. diagonālis, from Gr. διά, *through*, and γωνία, *angle*.

From a point F within the polygon, draw a straight line to each of the angles, dividing the polygon into as many triangles as it has sides.

The angles of each triangle = *hm* ∟? 54.

∴ The sum of the angles of all the triangles = *hm* ∟?

But the angles about *p*F = *hm* ∟? 29. 3.

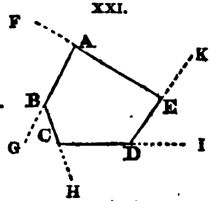
∴ The angles at A, B, C, D, and E, = *hm* ∟?

∴ The interior angles of the polygon are together = twice as many right angles as the polygon has sides, wanting how many?

§ 62. II. *Given*, any polygon ABCDE.

Required, the sum of its exterior angles.

XXI.



Each interior angle + its adjacent exterior = *hm* ∟? 50. *c*.

∴ All the interior angles + all the exterior = *hm* ∟?

But the interior angles = *hm* ∟? 61.

∴ The exterior angles = *hm* ∟?

To the angles at what point in Fig. XX. (§ 61) are the exterior angles of a polygon equal?

§ 63. Theor. VIII. ***The interior angles of any polygon are together equal to twice as many right angles, less four, as the figure has sides; and the exterior angles are equal to four right angles.***

[Proved by dividing the polygon into triangles, &c.]

a.) The interior angles of a quadrilateral = *hm* ∟? = *hm* °? of a pentagon? of a hexagon? of a heptagon? of an octagon? of a decagon? of a triangle (applying Theor. VIII.)?

How many sides has a polygon, if its interior angles are together equal to 4 ∟? if they are = 6 ∟? = 8 ∟? = 2 ∟? = 10 ∟? = 16 ∟? = 12 ∟?

In what kind of a polygon is the sum of the interior angles equal to the sum of the exterior? In what, is the sum of the interior angles half that of the exterior? In what, twice as great? In what, 3 times as great?

b.) In an equiangular triangle, how many degrees does each interior angle contain? each exterior angle? What fraction of a right angle is each?

Answer the same questions in respect to an equiangular quadrilateral, pentagon, hexagon, octagon, and decagon.

In equiangular polygons, as the number of sides increases, does each interior angle increase or diminish? each exterior angle?

In what kind of a figure is each interior angle twice as great as its adjacent exterior angle? In what, equal? In what, half as great? In what, 4 times as great?

c.) Can you fill up the space about a point with equiangular triangles? how many? with equiangular quadrilaterals? how many? with equiangular pentagons? with equiangular hexagons? how many?

Can you fill up the space about a point with equiangular polygons of any other kind? Why?

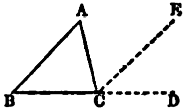
Which of these figures has the bee, that admirable geometer, adopted in the construction of its cell?

d.) If, in Fig. XXI. (§ 62), BAE = 70°, ABC = 140°, BCD = 110°, and CDE = 130°, *hm* ° does each of the other angles of the pentagon, both interior and exterior, contain?

If AEK = 100°, BAF = 90°, CBG = 60°, and DCH = 70°, *hm* ° does each of the other angles of the figure contain?

If BAF = 100°, BCD = 120°, EDI = 80°, and AED = 90°, *hm*° does each of the other angles of the figure contain?

XIX.



§ 64. *e.*) *a*ACD + *a*ACB = *hm* ∟? 50. *c.*

And *a*BAC + *a*ABC + *a*ACB = *hm* ∟? 54.

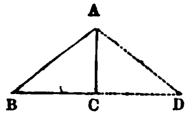
∴ ACD + ACB = BAC + ABC + ACB? 20. *a.*

∴ ACD = BAC + ABC? 20. *f.*

COR: *Any exterior angle of a triangle is equal to the sum of the two interior opposite angles.* It is of course greater than either singly.

Draw CE ∥ BA, and prove the same from Theor. V.

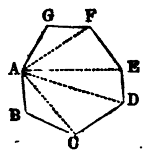
XXII.



Join AD; and then (as the angles of *t*ADC are together = the angles of *t*ADB) show, by subtracting the same angles from both sums, that *a*ACD = *a*BAC + *a*ABC.

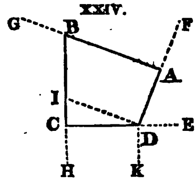
Prove from this Corollary, that the sum of the exterior angles of a triangle is equal to twice the sum of the interior angles.

XXIII.



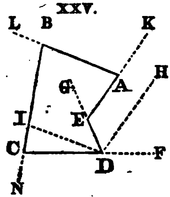
§ 65. *f.*) Divide a polygon into triangles, by drawing diagonals from one of the angles; and show, by this means, what is the sum of the interior angles.

XXIV.



§ 66. *g.*) From *p*D of the polygon ABCD, draw DI ∥ AG and DK ∥ BH; and then show (§ 46), that the exterior angles of the polygon are equal to the angles about D, and are, therefore, = 4 ∟.

XXV.



§ 67. *h.*) An interior angle of a polygon greater than 180° (convex inwardly, § 11. *b*), is termed *re-entrant*[h] (as, AED); and, in distinction, one less than 180° is termed *salient*[i].

It is evident, that a re-entrant angle cannot have, in the strict sense of the term, an *exterior* angle; since the

(h) Fr. rentrant, from Lat. re-, *again*, and intro, *to enter*. (i) L. saliens, *springing forth, projecting*.

angle made by one side with the other side produced falls *within* the figure. Still there is no need of excepting this case in the doctrine of exterior angles, if one distinction is only borne in mind; viz. that, while the exterior angle must be *added* to a *salient* angle to obtain 180°, it must be *subtracted* from a *re-entrant* angle to obtain the same. The exterior angle of a re-entrant angle is therefore a *negative quantity*, and must be subtracted in obtaining the sum of the exterior angles of a polygon.

§ 68. *i.*) That the exterior angles of every polygon must be equal to 4 right angles, is also evident from the consideration that, if we depart from the direction of CD (Fig. XXI. or XXIII.), we must pass through the whole circuit of divergence (§ 10), before we can come round to it again; and that it is indifferent how many bends we make in completing the circuit. Thus, in Fig. XXIII., the bend at D carries us from the direction of DE to that of DA; the bend at A, from that of DA to that of DI (‖ AG); the bend at B, from that of DI to that of DK (‖ BH); and the bend at C, from that of DK back to that of DE.

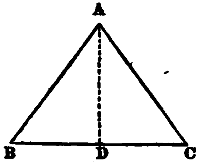
k.) In the case of a re-entrant angle, there is a turning back in the journey round, and a passing over the same ground twice. Thus (Fig. XXV.), after passing from the direction of DF to that of DE, we turn back at E to the direction of DH (‖ EK), and then proceed again at A to the direction of DI (‖ AL). But in every journey, the distance travelled back must, of course, be subtracted from the whole amount, to ascertain the onward progress.

If, in Fig. XXV., CBL = 100°, DCN = 90°, EDF = 120°, and AEG = 50°, *hm*° does BAK contain? *hm*° does each interior angle of the figure contain?

l.) A polygon which has no re-entrant angles is termed a *convex polygon*.

PROPOSITION IX.

XXVI.



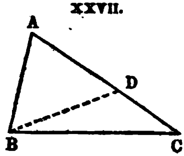
§ 69. I. *Given*, in *t*ABC, AC = AB.

Required, *a*B ≈ *a*C.

Draw AD bisecting[k] *a*A. Then, as the triangles ABD and ACD have AB = AC, *a*BAD = *a*CAD, and the side AD common, *t*ABD ≈ *t*ACD ? 40.

∴ *a*B ≈ *a*C ? 40. R.

XXVII.



§ 70. II. *Given*, in *t*ABC, AC > AB.

Required, *a*B ≈ *a*C.

In AC, take AD = AB, and join DB. Then, as AD = AB,

*a*ABD ≈ *a*ADB ? 60.

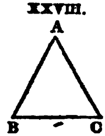
But ADB ≈ DCB (or ACB) ? 64.

∴ ABD ≈ ACB ? 20. *b*.

And ABC ≈ ABD ?

∴ ABC ≈ ACB ? 20. *c*.

XXVIII.



§ 71. III. *Given*, in *t*ABC, *a*B = *a*C.

Required, AC ≈ AB.

Must AC be either =, >, or < AB ?

If AC > AB, then *a*B ≈ *a*C ? 70.

Can, then, AC be > AB ? 23. *a*.

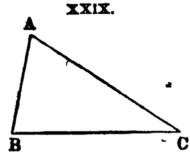
If AC < AB, then *a*B ≈ *a*C ?

(k) Lat. bis, *in two*, and seco, *to cut*, to divide into two equal parts.

Can, then, AC be $<$ AB?

$\therefore$ AC $\gtrless$ AB? 23. *b*.

XXIX.



§ 72. IV. *Given*, in *t*ABC, *a*B $>$ *a*C.

Required, AC $\gtrless$ AB.

If AC $=$ AB, then *a*B $\gtrless$ *a*C? 69.

If AC $<$ AB, then *a*B $\gtrless$ *a*C? 71.

$\therefore$ AC $\gtrless$ AB?

§ 73. THEOR. IX. ***In a triangle, equal sides subtend equal angles; greater, greater; and less, less.***

[Proved by dividing into identical triangles, reductio ad absurdum, &c.]

a.) If a triangle has two equal sides, how many equal angles has it? If it has two equal angles, how many equal sides has it?

If the sides of a triangle are all equal, how do its angles compare with each other? If its angles are all equal, how do its sides compare with each other?

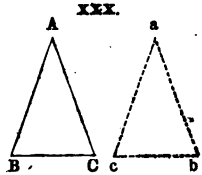
If the sides of a triangle are all unequal, how do its angles compare with each other? If its angles are all unequal, how do its sides compare with each other?

What equal parts has an isosceles triangle? an equilateral triangle? a scalene triangle?

In the general proposition above are included the following particulars:

(1.) *The angles at the base of an isosceles triangle are equal.* (2.) *A triangle which has two equal angles is isosceles.* (3.) *An equilateral triangle is also equiangular.* (4.) *An equiangular triangle is also equilateral.* (5.) *In a triangle, the greater of two unequal sides subtends a greater angle;* and (6.) *the greater of two unequal angles is subtended by a greater side.*

XXX.



§ 74. *b.*) Nos. 1 and 2 above may be also shown thus :

(1.) *Given*, in *t*ABC, AB = AC.

Let *t*abc represent *t*ABC inverted.

As the two triangles have the same angle at the vertex, and the adjacent sides all equal, would they coincide, if applied to each other, AB to ac, and AC to ab? 39. I.

∴ aB ≈ c (aC inverted)?

(2.) *Given*, aB = aC.

As *t*ABC and *t*abc (*t*ABC inverted) have the same base, and the adjacent angles all equal, would they coincide, if applied to each other, B to c, and C to b? 39. II.

∴ AB ≈ ac (AC inverted)?

§ 75. *c.*) If the legs of an isosceles triangle are produced, how do the exterior angles compare with each other? How do the exterior angles of an equilateral triangle compare with each other? of a scalene triangle?

Each interior angle of an equilateral triangle = *hm*°? Each exterior angle?

To which class of angles (right, obtuse, or acute) do the equal interior angles of an isosceles triangle belong? The equal exterior angles?

d.) If a right triangle (§ 55. *c*) is isosceles, *hm*° does each interior angle contain? each exterior?

If the angle at the vertex of an isosceles triangle is 80°, *hm*° does each angle at the base contain? each exterior angle of the triangle? If the angle at the vertex is 40°?

If one of the angles at the base of an isosceles triangle is 40°, *hm*° does the angle at the vertex contain?

In an isosceles triangle, if the angle at the vertex is given, how do you find one of the angles at the base? If one of the angles at the base is given, how do you find the angle at the vertex?

An interior angle at the base of an isosceles triangle is what part of the exterior angle at the vertex?

e.) In a right triangle, which is the longest side? Why? In an obtuse triangle, which is the longest side?

COR. I. *In a right triangle, the hypotenuse is always the longest side.*

§ 76. *f.*) As the triangles ABD and ACD (Fig. XXVI., § 69) are identical, *a*ADB ≋ *a*ADC?

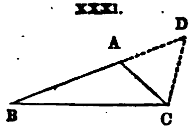
∴ AD $\wr$ BC? 26.

And BD ≋ DC?

COR. II. *A straight line bisecting the angle at the vertex of an isosceles triangle is perpendicular to the base and bisects it.*

PROPOSITION X.

XXXI.



§ 77. *Given*, any triangle ABC.

Required, any one side ≋ the sum of the other two.

Let BC be the longest side, or at least not less than either of the others. Produce BA to D, making AD = AC; and join DC.

As AC = AD, *a*ADC ≋ *a*ACD? 72. 1.

∴ ADC ≋ BCD?

∴ In *t*DBC, BC ≋ BD? 73.

But, as AD = AC, BD ≋ BA + AC?

∴ BC ≋ BA + AC?

§ 78. THEOR. X. *Any side of a triangle is less than the sum of the other two.*

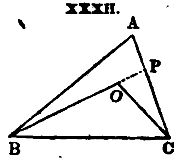
[Proved by the aid of Theor. IX.]

§ 79. *a.*) As BC < BA + AC, BC — AC ≋ BA? 20. *g.*

And BC — BA ≋ AC?

Cor. i. *The difference between any two sides of a triangle is less than the third side.*

XXXII.



§ 80. *b.*) *Given*, OB and OC, drawn from any point O within *t*ABC, to the extremities of one side BC.

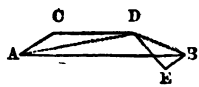
Required, BO + OC ≋ BA + AC; and *a*BOC ≋ *a*BAC.

Produce BO to P. Then OC ≋ OP + PC ? 78.
∴ Adding BO, BO + OC ≋ BP + PC ?
But BP ≋ BA + AP ?
∴ Adding PC, BP + PC ≋ BA + AC ?
∴ BO + OC ≋ BA + AC ? 20. *c.*

Again, *a*BOC ≋ *a*OPC ? 64.
And, OPC ≋ BAC ?
∴ BOC ≋ BAC ?

Cor. ii. *Two straight lines drawn from a point within a triangle to the extremities of any side are together less than the other two sides, but contain a greater angle.*

XXXIII.



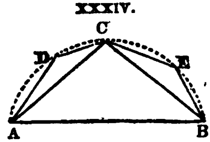
§ 81. *c.*) *Given*, between the two points A and B, the straight line AB, and any broken line ACDEB.

Required, AB ≋ ACDEB.

Join AD and DB. Then AB ≋ AD + DB ? 78.
But AD ≋ AC + CD ? and DB ≋ DE + EB ?
∴ AB ≋ ACDEB ?

Cor. iii. *Between two points, a straight line is shorter than any broken line.*

XXXIV.



§ 82. *d.*) Draw a straight line between A and B. Draw a line between A and B, having a single bend at C. Then *l*ACB ≋ AB ?

How would the length of the line between A and B be affected by an

additional bend between A and C, as at D? by an additional bend between C and B, as at E? by still additional bends between A and D, D and C, C and E, E and B? by any additional bend that should be made in like manner between angular points that remain fixed?

Suppose the process continued, until the line bends at every point, or, in other words, becomes a curve (§ 5) as the dotted line ADCEB. Has each additional bend increased the length of the line?

How, then, does the curve ADCEB compare in length with the straight line AB? with the broken line ACB? with the broken line ADCB? with the broken line ADCEB? with each broken line that is formed by the continued process? with any broken line between A and B whose angular points should all be in the curve?

Cor. iv. *Between two points, any curve line is longer than a straight line, and longer than any broken line whose angular points are all in the curve.*

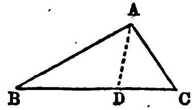
§ 83. *e.*) Cor. v. It follows from §§ 81 and 82, that

A STRAIGHT LINE *is the shortest distance between two points.*

Hence, a straight line is the *measure of distance* (sometimes termed simply *the distance*) between two points.

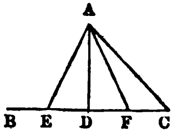
§ 84. *f.*) Prove Theor. X. by bisecting *a*BAC, and by showing (since BDA > DAC, and, therefore, > BAD) that BA > BD, and in like manner, that AC > DC (§§ 64, 73).

XXXV.



Proposition XI.

XXXVI.



§ 85. *Given*, between A and BC, AD ⊥ BC, AE and AF meeting BC at equal distances from D, and AC meeting BC at a greater distance from D than AE or AF.

Required, AD ≋ AE ≋ AF ≋ AC.

(1.) In the right triangle ADE, *a*AED ≋ *a*ADE ?

∴ AD ≋ AE ? 73, 76. *e*.

(2.) Can you show that the triangles ADE and ADF are identical ? 39. 1.

∴ AE ≋ AF ?

(3.) In *t*AFC, *a*ACF ≋ *a*AFC ? Why ?

∴ AF (or its equal AE) ≋ AC ?

§ 86. Theor. XI. *Of the lines that can be drawn to a given straight line from a point without,*

(1.) *The perpendicular is the shortest.*

(2.) *Of oblique lines, those which meet the given line at equal distances from the perpendicular are equal ; and* (3.) *those which are nearer the perpendicular are shorter than those which are more remote.*

[Proved by applying Theor. IX., &c.]

§ 87. *a.*) Hence a perpendicular is the *measure of distance* between a point and a straight line. Compare § 83.

§ 88. *b.*) The converse of this Theorem may be readily shown ; thus,

(1.) If AD is the shortest line that can be drawn from A

to BC, it is $\perp$ BC; for, otherwise, a perpendicular could be drawn which would be shorter.

(2.) If AE = AF, then DE = DF; for, if DE were $>$ or $<$ DF, AE must be $>$ or $<$ AF.

(3.) If AE $<$ AC, then DE $<$ DC; for if DE were = or $>$ DC, AE must be = or $>$ AC.

§ 89. *c.*) Having drawn, from A to BC, the perpendicular AD, draw any other line AE. Show, by comparing the angles ADE and AED, that AE cannot be $\perp$ BC. See § 50.

Show the same, by comparing the exterior angle AEB with the interior opposite angle ADE.

How many lines can be drawn from A to BC, making the two adjacent angles equal?

As an oblique line between A and BC is drawn farther from the perpendicular, does it make the adjacent angles more or less unequal (or, in other words, does it meet BC more or less obliquely)?

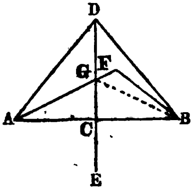
§ 90. *d.*) Besides AE, can you draw another line from A to BC, equal to AF? How many lines can you draw from A to BC, equal to AC?

Can two lines drawn from A to BC upon the same side of AD be equal to each other?

Can, then, three equal lines be drawn from A to BC?

COR. I. *Of the straight lines that can be drawn to a given straight line from a point without,* — (1.) *no three can be equal; and* (2.) *any two that are equal must be upon opposite sides of the perpendicular.*

XXXVII.



§ 91. *e.*) *Given,* any point D in DE drawn $\perp$ AB through C, the middle point of AB; and any point F, not in DE.

Required, DA ≋ DB; and FA ≋ FB.

As DA and DB meet AB at equal distances from the perpendicular DC, DA ≋ DB? 86. 2.

From G (where FA crosses the perpendicular), draw GB.

Then, FG + GB ≋ FB ? 78.

But GB ≋ AG ?

∴ FG + GA (or FA) ≋ FB ?

COR. II. *If through the middle point of a given straight line a perpendicular be drawn, any point in that perpendicular will be equally distant from the two extremities of the given line; and any point without the perpendicular will be unequally distant.*

§ 92. *f.*) *Given*, in DE (Fig. XXXVII.), any two points D and G, each equally distant from the two extremities of AB.

Required, DE ‖ AB; and AC ≋ AB.

If no point not in the perpendicular drawn through the middle of AB can be equally distant from A and B, then in what line must the points D and G both lie? 91.

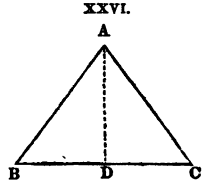
Can any other straight line pass through them both, except DE? 6. *a.*

∴ DE ‖ AB ? and AC ≋ CB ?

COR. III. *If any two points of one straight line are each equally distant from the extremities of a second, the first line will bisect the second at right angles.*

g.) Hence, if one straight line passes through the middle of a second, and any other point of the first is equally distant from the extremities of the second, what angles do the two lines make with each other?

XXVI.



§ 93. *h.*) In an isosceles triangle,

(1.) How does a perpendicular drawn from the vertex to the base divide the base? 88. 2.

(2.) What angles does a line drawn from the vertex to the middle of the base make with the base? 92. *g.*

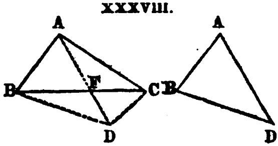
(3.) If a perpendicular be raised from the middle of the base, will it pass through the vertex? Why? 91.

In any triangle not isosceles, will a perpendicular raised from the middle of the base pass through the vertex?

If ADB and AGB (Fig. XXXVII.) are two isosceles triangles upon the same base AB, how will the straight line DE passing through their vertices divide the base? at what angles?

Proposition XII.

XXXVIII.



§ 94. I. *Given*, in two triangles ABC and ABD, AB = AB, and AC = AD, but *a*BAC > *a*BAD.

Required, BC ≋ BD.

Of the two sides AB and AC, let AB be that which is not longer than the other. Then place *t*ABD upon *t*ABC with AB upon AB.

As AB = or < AC,	*a*ACB ≋ *a*ABC?	73.
But	AFC ≋ ABC?	64.
∴	AFC ≋ ACF?	
∴	AC ≋ AF?	73.

As AD = AC, must *p*D fall without *t*ABC?

Join CD.	Then, as AC = AD, *a*ADC ≋ *a*ACD?	73. *a*.
But	BDC ≋ ADC, and ACD ≋ BCD?	
∴	BDC ≋ BCD?	20. *c*.
∴	BC ≋ BD?	73.

§ 95. II. *Given*, in *t*ABC and *t*ABD (Fig. XXXVIII.), AB = AB, and AC = AD, but BC > BD.

Required, *a*BAC ≋ *a*BAD.

	If BAC = BAD, then BC ≋ BD?	40.
	And if BAC < BAD, then BC ≋ BD?	94.
∴	BAC ≋ BAD?	23. *b*.

§ 96. THEOR. XII. *In triangles agreeing in two sides, the greater included angle is subtended by the greater third side.*

[Proved by superposition with the aid of Theor. IX., and by reductio ad absurdum.]

§ 97. REMARK. In § 94, we might proceed thus, without joining CD:

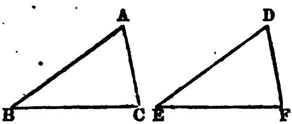
AF + CF ≋ AC, and FD + FB ≋ BD? 78.

∴ AD (= AF + FD) + BC (= CF + FB) ≋ AC + BD? 20. *e.*

∴ As AD = AC, BC ≋ BD? 20. *g.*

PROPOSITION XIII.

IX.



§ 98. I. *Given*, two triangles ABC and DEF agreeing in the three sides; viz. AB = DE, AC = DF, and BC = EF.

Required, *t*ABC ≋ *t*DEF.

If *a*A > *a*D, then BC ≋ EF? 96.

If *a*A < *a*D, then BC ≋ EF?

∴ *a*A ≋ *a*D? 23: *b*.

Can you now prove that the two triangles are identical? 89. I.
Name each pair of equal angles.

A.) *Triangles agreeing in the three sides are identical.*

§ 99. II. *Given*, two triangles ABC and DEF (Fig. IX., § 98) agreeing in two angles and the side opposite to one of them; viz. *a*A = *a*D, *a*B = *a*E, and BC = EF.

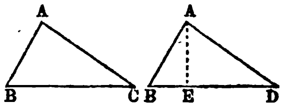
Required, *t*ABC ≋ *t*DEF.

As $aA = aD$, and $aB = aE$, $aC \approx aF$? 54. c.

∴ $tABC \approx tDEF$? 39. II.

B.) *Triangles agreeing in two angles and the side opposite to one of them are identical.*

XXXIX.



§ 100. III. *Given*, two triangles ABC and ABD agreeing in two sides and the angle opposite to one of them; viz. $AB = AB$, $AC = AD$, and $aABC = aABD$.

Required, $tABC \approx tABD$.

Place $tABC$ upon $tABD$ with AB upon AB. As $aABC = aABD$, where will BC fall?

From A draw AE ⊥ BD.

(1.) Let AC (= AD) be not less than AB. Then can either AC or AD fall between AB and AE? 86. 3.

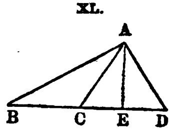
Must, then, AC and AD both fall upon the same side of AE? As AC is equal to AD, can it fall anywhere else except upon AD? 86, 90.

Where must C fall?

∴ $tABC \approx tABD$? 17. a.

(2.) Let AC (= AD) be < AB.

XL.



Can AC or AD now fall between AB and AE? 86.

If both AC and AD fall upon the same side of AE, must they form the same line? 86, 90.

Are the given triangles in this case identical?

If AC and AD fall upon opposite sides of AE, as in the figure above, then $tABC \approx tABD$?

As AC = AD, what kind of a triangle forms the difference between $tABD$ and $tABC$?

What kind of an angle is ACB? ADB?

If AC and AD should be = AE, where would C and D then fall? 86, 89.

Would the triangles in this case be identical? What kind of triangles would they be?

C.) *Two triangles agreeing in two sides and the angle opposite to one of them are identical; except when the given angle is opposite to the less of the two given sides, and the angle opposite to the greater side is obtuse in one of the triangles and acute in the other.*

a.) Can a right angle ever be opposite to the less of two sides of a triangle? 75. *e.*

Can, then, the uncertainty above ever arise in a right triangle?

Right triangles agreeing in any two like sides (§ 58) *are identical.*

b.) Can the uncertainty ever arise, when the given angle is obtuse?

§ 101. We have now considered (§§ 39, 40, 98 – 100) all the cases in which triangles agree in *three similarly situated and independent parts* (meaning by *independent parts* those which are not determined by others that are given, and of course excluding from the number the third angle, as it is determined by the other two, § 54). The result may be expressed in the following general theorem, which, however, must be received with the qualification above stated (§ 100. C).

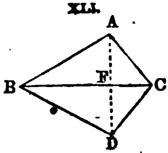
THEOR. XIII. ***Triangles agreeing in any three similarly situated and independent parts are identical.***

§ 102. *a.*) If triangles agree in the three angles, can you determine from this that they are identical?

If simply the three angles of a triangle are given, can you determine any thing in respect to the size of the triangle.

b.) The following direct mode of proof might have been employed in § 98.

XLI.



In the triangles ABC and DBC, let AB = DB, AC = CD, and BC (a side not less than either of the others) = BC.

Unite the triangles as in the figure, and join AD. Show (as BC is not less than AB or AC), that AD must fall between B and C.

As AB = DB, *a*BAD ≋ *a*BDA ? 69.

As AC = CD, *a*CAD ≋ *a*CDA ?

∴ *a*BAC ≋ *a*BDC ? 20. *d.*

∴ *t*BAC ≋ *t*BDC ?

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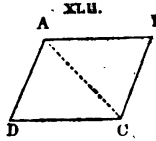
§ 103. Definition. A quadrilateral of which the opposite sides are parallel is termed a **parallelogram**[1].

Proposition XIV.

XLII.



§ 104. I. *Given*, any parallelogram ABCD.

Required, AB ≋ DC, AD ≋ BC, *a*B ≋ *a*D, and *a*A ≋ *a*C.

Join AC. Then, as AB ∥ DC, *a*BAC ≋ *a*DCA ? 46. *c.*

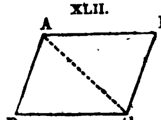
As BC ∥ AD, *a*BCA ≋ *a*DAC ?

∴ As AC is common, *t*BAC ≋ *t*DCA ?

∴ AB ≋ DC, AD ≋ BC, and *a*B ≋ *a*D ?

And *a*A (= BAC + DAC) ≋ *a*C (= DCA + BCA) ? 20. *d.*

XLII.



§ 105. II. *Given*, in the quadrilateral ABCD, AB = DC, and AD = BC.

Required, AB ∥ DC, and AD ∥ BC ?

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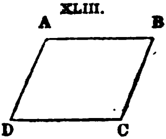
(1) Gr. παραλληλόγραμμος, *bounded by parallel lines*, from παράλληλος, *parallel*, and γραμμή, *line*.

Join AC. Can you then show that the triangles BAC and DCA are identical? · 99.

∴ *a*BAC ≋ *a*DCA? ∴ AB ∥ DC? 49. *c*.

And *a*DAC ≋ *a*BCA? ∴ AD ∥ BC?

XLIII.



§ 106. III. ***Given***, in the quadrilateral ABCD, *a*A = *a*C, and *a*D = *a*B.

Required, AB ∥ DC, and AD ∥ BC.

*a*A + *a*D + *a*C + *a*B = *hm*∟? 63. *a*.

∴ *a*A + *a*D (= *a*C + *a*B) = *hm*∟?

∴ AB ∥ DC? 40. *e*.

And *a*A + *a*B = *hm*∟? ∴ AD ∥ BC?

§ 107. IV. ***Given***, in the quadrilateral ABCD (Fig. XLII.), AB = and ∥ DC.

Required, BC ≋ and ∥ AD.

Join AC. Then, as AB ∥ DC, *a*BAC ≋ *a*DCA? 48. *c*.

∴ As AB = CD, and AC = CA, *t*BAC ≋ *t*DCA? 40.

∴ BC ≋ AD?

And *a*ACB ≋ *a*CAD? ∴ BC ∥ AD?

§ 108. Theor. XIV. (1.) ***The opposite sides and angles of a parallelogram are equal; and*** (2.) ***any quadrilateral of which the opposite sides or*** (3.) ***the opposite angles are equal, or*** (4.) ***of which two sides are equal and parallel, is a parallelogram.***

How many pairs of equal parts does each parallelogram contain?

§ 109. *a*.) The two angles of a parallelogram adjacent to any side = *hm*∟? = *hm*°? 46. *e*.

If, then, one angle of a parallelogram is a right angle,

what must the other angles be? If one is acute? If one is obtuse?

Must a parallelogram be either right-angled throughout, or oblique-angled throughout?

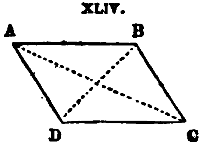
If any two adjoining sides of a parallelogram are equal, how do all the sides compare with each other?

Cor. I. *In a parallelogram, if one of the angles is a right angle, all are right angles; and if two adjoining sides are equal, the sides are all equal.*

b.) If the angles of a parallelogram are all equal, what kind of an angle must each be?

If, in an oblique parallelogram, each obtuse angle is double each acute, *hm*° does each angle contain? If each obtuse angle is triple each acute?

XLIV.



§ 110. *c.*) How many diagonals (§ 60) can be drawn in a parallelogram?

How do the triangles into which a diagonal divides a parallelogram compare with each other? 104.

What parts have *t*ADC and the parallelogram ABCD in common? *t*BAD and ABCD?

If a triangle and parallelogram have two sides and the included angle common, what part is the triangle of the parallelogram?

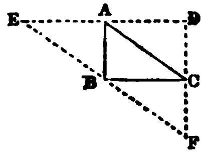
Can a triangle and parallelogram have a side and two adjacent angles common?

If a triangle and a parallelogram agree in three adjacent parts, what must those parts be? How does the triangle then compare with the parallelogram in extent of surface, or area (§ 16. *b*)?

Cor. II. *A diagonal divides a parallelogram into two identical triangles. Each of these has two sides and the included angle in common with the parallelogram.*

§ 111. *d.*) How many parallelograms can be drawn having three adjacent parts in common with a given triangle ABC?

XLVI.



Show what lines are drawn, and how, to complete each parallelogram.

What parts are common to *t*ABC and the parallelogram EC? to ABC and DB? to ABC and AF?

What part is common to EC and DB? to EC and AF? to DB and AF?

EA $\approx$ AD? DC $\approx$ CF? EB $\approx$ BF? 108, 20. *a.*

How do the parallelograms compare with each other in area? 20. *h.*

What part is ABC of each parallelogram?

How many identical triangles does the figure above present?

If AB = 3 ft., BC = 4 ft., and AC = 5 ft., what are the dimensions of all the sides of each parallelogram? What is the perimeter (§ 36) of each parallelogram?

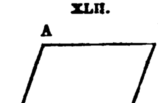
If AB = 6 ft., BC = 8 ft., and AC = 10 ft., what is the perimeter of each parallelogram?

If *a*ABC = 90°, and *a*BAC = 50°, *hm*° does each angle of each parallelogram contain? How do all the angles of BD compare with each other?

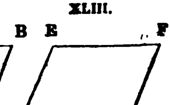
If BAC = 60°, and ACB = 40°, *hm*° does each angle of each parallelogram contain? If ABC = 90°, and ABF = 125°?

§ 112. *e.*) If two parallelograms (as AC, EG) agree in one angle (e. g. D = H) and the including sides, show by superposition that they are identical.

XLII.



XLIII.



§ 113. *f.*) Show directly from Theor. V., that the angles of every parallelogram are together = 4 ∟.

Can every triangle with an identical triangle form a parallelogram? 111.

What new proof do you find here, that the angles of a triangle are together = 2 ∟?

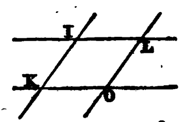
g.) Are the opposite angles of a parallelogram like or unlike angles about parallels?

Is their equality, then, directly established by Theor. V.?

h.) The construction of the common *parallel ruler* depends upon § 108. 2.

§ 114. *i*.) When two parallels are intercepted[m] between two other parallels (as IK and LO between IL and KO, or IL and KO between IK and LO), what species of quadrilateral is formed? 108.

XIV.

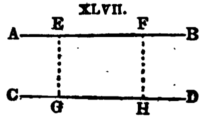


∴ IK ≋ LO, or IL ≋ KO? 108.

Cor. III. *Two parallels intercepted between two other parallels are equal.*

§ 115. *k*.) In AB parallel to CD, take any two points E and F, and from them draw EG and FH perpendicular to CD. Then EG ∥ FH? 49. *a*.

XLVII.



∴ EG ≋ FH? 114.

What line measures the distance of *p*E from CD? of *p*F from CD? of *p*G from AB? of *p*H from AB? 87.

How, then, do E and F compare in respect to distance from CD? How do G and H compare in respect to distance from AB?

Could the same be shown in respect to any other points taken in AB or CD?

Cor. IV. *Two parallels are everywhere equidistant*[n]; *and any perpendicular between two parallels is a measure of their distance.*

(m) Lat. interceptus, *taken in between*. (n) L. æquidistans.

Can parallels ever approach or recede from each other, converge or diverge?

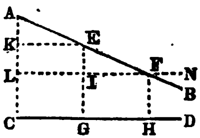
§ 116. *l.*) Prove from Cor. IV., that *only one parallel to a given line can pass through a given point.*

Prove the same from § 7 and § 6. *b.*

§ 117. *m.*) If EG and FH (Fig. XLVII.) are given perpendicular to CD and equal to each other, then AB ∦ CD? 108.

Cor. v. *If any two points in one straight line are equidistant from another straight line, the two lines are parallel.*

XLVIII.



§ 118. *n.*) In AB inclined to CD (§ 7), take any two points A and E, and make EF = AE. From A, E, and F, draw AC, EG, and FH perpendicular to CD; and from E and F, draw EK and FIL parallel to DC.

As EK ∥ FL, *a*AEK ≈ *a*EFI? 46. *b.*

And, as AC ∥ EG, *a*EAK ≈ *a*FEI?

∴ As AE = EF, *t*AKE ≈ *t*EIF, and AK ≈ EI? 40.

But, since EK, FL, and DC are parallel, and AC, EG, and FH are also parallel, KC ≈ EG, LC ≈ IG ≈ FH, and KL ≈ EI?

What line represents the distance of A from CD? What part of AC represents the distance of E from CD? of F? What represents the difference between the distance of A from CD and that of E? between the distance of E and that of F?

Are these differences equal?

Do you, then, make equal approaches to CD in going from A to E, and in going the equal distance from E to F? On the other hand, do you recede equally from DC in going from F to E, and from E to A?

If any two equal distances were taken upon CD, could you show, in like manner, that in these equal distances you approach or recede from AB equally?

Do, then, AB and CD approach each other on the one side, and recede from each other on the other side, uniformly (i. e. equally in equal distances)?

Cor. vi. *Inclined straight lines approach or recede from each other uniformly.*

§ 119. *o.*) If AK (Fig. XLVIII.) be a fourth part of AC, at what distance from A (i. e. at how many times the distance AE) will AB and CD meet? If AK $= \frac{1}{5}$ AC? $\frac{1}{8}$ AC? $\frac{1}{100}$ AC? $\frac{1}{1000}$ AC?

Whatever part AK is of AC, can you so multiply AE, that AB and CD will at length meet?

On the other side, is there any limit to the distance to which they will recede from each other if produced?

Cor. vii. *If inclined straight lines are produced, they will meet on the one side, and will recede from each other without limit on the other side.*

p.) If AB will meet CD, will all lines parallel to AB meet all lines parallel to CD lying in the same plane?

§ 120. *q.*) In Fig. XLVIII., aBFH + aFHD ≈ 2∟?
And aAFH + aFHC ≈ 2∟?

If two straight lines make, with a third, the interior angles upon the same side greater or less than two right angles, are the two lines parallel or inclined? 48.

If produced, will they meet on the side on which the two interior angles are greater or less than two right angles?

As these interior angles differ more from 2∟, will the lines approach more or less rapidly?

As the interior angles upon the other side differ more from 2∟, will the lines recede from each other more or less rapidly?

In Fig. XLVIII., aBFH ≈ aFHC?
And aAFH ≈ aFHD?

How do the alternate-interior angles (§ 42. *a*) which two inclined straight lines make with a third line, compare with each other?

Will the two lines meet, if produced, upon the side of the greater or the less of these angles?

As the inequality of these angles is greater, will the two lines converge (or diverge) more or less rapidly?

r.) What angles in Fig. XLVIII. are equal to the angle which AB and CD would make with each other at the point of meeting?

NOTE. Lines not actually meeting are spoken of as making with each other the angle which they would make if produced; and this angle may be obtained either by producing the lines, or by drawing from any point in one a line parallel to the other. See § 9. *d.*

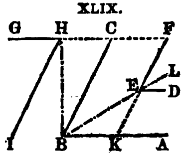
s.) In a right-angled triangle, if you diminish the angle at the base, what effect is produced upon the perpendicular? if you increase it?

If you diminish the angle at the vertex, what effect is produced upon the base? if you increase it?

Continue to increase the angle at the vertex until it becomes a right angle, what effect is then produced?

If you increase the perpendicular, what effect is produced upon the angle at the base? upon the angle at the vertex? If you increase the base?

XLIX.



§ 121. *t.*) *Given*, two angles ABC and DEF, having the two sides of the one parallel to the two sides of the other in the same order (i. e. as you pass round, the first side of the one being parallel to the first side of the other, and the second to the second); viz. BA ∥ ED, and BC ∥ EF.

Required, ABC ≈ DEF.

If those sides of the two angles which are not parallel do not already meet, will they meet if produced?

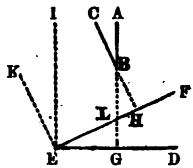
Let FE meet AB at K. Then, ABC ≈ AKE, and AKE ≈ DEF? 46. *b.*

∴ ABC ≈ DEF?

Producing GH to C, or to F, compare in like manner GHI (having GH ∥ BA, and HI ∥ BC) with ABC or DEF.

u.) Instead of producing the sides of the angles in Fig. XLIX., join their vertices by drawing BEL and HB; and then prove, by adding equals to equals, or by subtracting equals from equals, that ABC is equal to DEF, and to GHI.

L.



§ 122. *v.*) *Given*, two angles ABC and DEF, having the two sides of the one alike inclined to the two sides of the other in the same order; viz. AB having the same inclination to DE which CB has to FE.

Required, ABC ≋ DEF.

Let AB meet DE at G; and let CB meet FE at H.

Then by hypothesis, CHF ≋ AGD? 9. *d*, 120. *r*.

Draw IE ∥ AG, and KE ∥ CH. Then IEK ≋ ABC? 121.

As KE ∥ CH, KEH ≋ CHF? 46. *b*.

As IE ∥ AG, IED ≋ AGD?

∴ KEH ≋ IED?

Subtracting IEH common to both, IEK ≋ DEF?

∴ ABC ≋ DEF?

Can you show the same by a comparison of the angles of the triangles ELG and BLH?

Hence (§§ 121, 122),

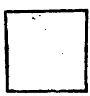
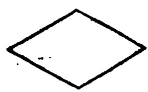
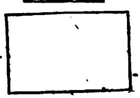
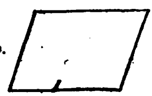
§ 123. Cor. viii. *Two angles are equal, if the sides of the one are parallel or alike inclined to the sides of the other in the same order.*

Notes. (1.) This Corollary might be illustrated and demonstrated in the same manner as Theor. V. in §§ 51, 52. (2.) The limitation "*in the same order*" is essential, for it will be observed that, if the order is not the same (as in ABC and DEK, Fig. XLIX.; or in DEF and CBL, Fig. L.), then the angles, instead of being *equal*, are *supplementary*.

§ 124. **Definitions.** **Parallelograms** are divided into two classes with respect to their angles (§ 109) : I. Right-angled Parallelograms, or Rectangles[o] ; and II. Oblique-angled Parallelograms. Each of these classes is divided into two species with respect to their sides : 1. *Those which are equilateral* ; and 2. *Those which are not equilateral.*

An *equilateral rectangle* is called a Square[p] ; a *rectangle which is not equilateral*, an Oblong[q], or oftener simply a *rectangle*. An *oblique parallelogram* (§ 55. *c*), *if equilateral*, is called a Rhombus[r] or Lozenge[s] ; *if not equilateral*, a Rhomboid[t].

PARALLELOGRAMS.

		Class I. RECTANGLES.		Class II. OBLIQUE PARALLELOGRAMS.
Species I. Equilateral.	Square.		Rhombus.	
Species II. Not Equilateral.	Oblong.		Rhomboid.	

Notes. *a.*) A quadrilateral is called a Trapezoid[u], when two of its sides are parallel ; and a Trapezium[v], when no two are parallel.

(o) Lat. rectus, *right*, and angulus, *angle*. (p) L. quadra, Fr. quarré. (q) L. oblongus. (r) Gr. ῥόμβος, *rhomb*, *top*, from ῥέμβω, *to whirl*. (s) Fr. losange. (t) Gr. ῥομβοειδής, *rhombus-like*. (u) Gr. τραπεζοειδής, *table-like*, from τράπεζα, *table*, and εἶδος, *form*. (v) Gr. τραπέζιον, *small* (or *ill-shaped*) *table*. There is a want of uniformity in the use of the terms *trapezoid* and *trapezium*.

b.) In a parallelogram, any side may be taken as a *base* (§ 56). In a trapezoid, the parallel sides are regarded as bases (the one *superior*[w], and the other *inferior*[x]).

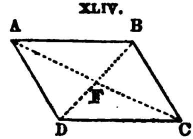
c.) The *altitude*[y], or *height*, of a figure is the distance to the base from the opposite vertex or side. It is measured by a perpendicular let fall upon the base (§§ 87, 115). Hence, in a rectangle, and commonly in a right triangle (§ 58), one of the sides is the measure of the altitude.

d.) By *the square of a line*, is meant the square of which the line is one side. By *the rectangle of two lines*, or *of one line into another*, is meant the rectangle of which the two lines are adjoining sides. The rectangle is also said to be *contained* by these sides.

e.) The rectangle of two lines which are equal is, of course, the same with the square of either.

Proposition XV.

XLIV.



§ 125. I. *Given*, any parallelogram ABCD with its diagonals intersecting at F.

Required, AF ≋ FC, and BF ≋ FD.

AB ≋ CD? 108.

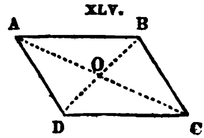
As AB ∥ CD, *a*ABF ≋ *a*CDF, and *a*BAF ≋ *a*DCF? 46. *c.*

∴ *t*ABF ≋ *t*CDF? 40.

∴ AF ≋ FC, and BF ≋ FD?

Show the same by comparing the triangles AFD and CFB.

XLV.



§ 126. II. *Given*, in the quadrilateral ABCD, the diagonals AC and BD bisecting each other in O.

(w) Lat., *upper*. (x) Lat., *lower*. (y) L. altitudo, *height*.

Required, AB ⋕ DC, and AD ⋕ BC.

What parts of *t*AOB = like parts of *t*COD?

∴ *t*AOB ≋ *t*COD? ∴ AB ≋ DC? 40.

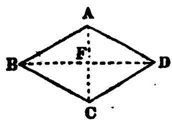
And *a*BAO ≋ *a*DCO? ∴ AB ⋕ DC? 49. *c*.

As AB = and ‖ DC, AD ≋ and ⋕ BC? 107.

Or show by comparing *t*AOD with *t*COB that AD is = BC; and that, consequently, ABCD (having its opposite sides equal) is a parallelogram. 108. 2.

§ 127. THEOR. XV. ***The diagonals of a parallelogram bisect each other; and any quadrilateral in which the diagonals bisect each other is a parallelogram.***

LI.



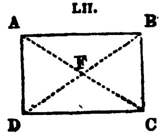
§ 128. *a*.) If ABCD be a square or rhombus, as AB = AD, and BF = FD, AC ⋕ BD? 92.

What kind of triangles are AFB, AFD, BFC, and DFC? 55.

How do these triangles compare with each other?

COR. I. *In an equilateral parallelogram the diagonals are perpendicular to each other, and divide the figure into identical triangles.*

LII.



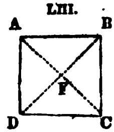
§ 129. *b*.) If ABCD be a rectangle, what three parts of *t*BAD = three like parts of *t*CDA?

∴ BD ≋ AC? ∴ BF ≋ AF?

What kind of a triangle, then, is FAB? FDC? FAD? FBC?

COR. II. *In a rectangle, the diagonals are equal, and divide the figure into isosceles triangles.*

LIII.



Describe the triangles into which a square is divided by its diagonals.

§ 130. *c*.) Prove the converse of Cor. I., that *a parallelogram whose diagonals cross at right angles is equilateral.*

d.) Prove the converse of Cor. II., that *a parallelogram whose diagonals are equal is a rectangle.*

§ 131. *e.*) If BAD (Fig. LII.) is a right angle, then AF, drawn from A to the middle of the hypotenuse BD, = what part of BD?

Conversely, if, in *t*BAD, AF, drawn from A to the middle of the opposite side BD, is = $\frac{1}{2}$ BD, then *a*BAD = *hm*°?

Hence, as every right triangle is the half of a rectangle (§§ 111, 113. *f*),

COR. III. *In a right triangle, the middle of the hypotenuse is equidistant from each angle; and conversely, any side of a triangle whose middle is equidistant from each angle subtends a right angle.*

Into triangles of what kind is a right triangle divided by a straight line drawn from the right angle to the middle of the hypotenuse?

§ 132. *f.*) Cor. III. may be also proved as follows:

(1.) If BAD (Fig. LII.) is a right angle, then *a*ABF + *a*ADF $\gtreqless$ *a*BAD? 54.

But, if AF > FB (or its equal FD), then ABF $\gtreqless$ BAF, and ADF $\gtreqless$ FAD? 73.

Then ABF + ADF $\gtreqless$ BAD?

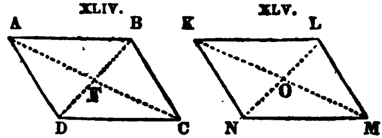
And, if AF < FB (or FD), then ABF + ADF $\gtreqless$ BAD?

∴ AF $\gtreqless$ FB (or FD)? 23. *b.*

(2.) If AF = FB = FD, then *a*ABF $\gtreqless$ *a*BAF, and *a*ADF $\gtreqless$ *a*FAD? 73.

∴ ABF + ADF $\gtreqless$ BAD? ∴ BAD = *hm*°?

XLIV. XLV.



§ 133. *g.*) If, in two parallelograms, the diagonals are equal, each to each, and cross at the same angles, prove that the parallelograms are identical.

PART THIRD.

PROBLEMS.

§ 134. Remarks. 1. The only instruments required for the following Problems are a Ruler for drawing straight lines, and Compasses or Dividers for taking or laying off distances, and for drawing arcs of circles. The beginner may sometimes need directions from his teacher in regard to the best method of holding or applying these instruments. If the Ruler is not at hand, a straight edge may be obtained by doubling over a piece of paper upon itself; and by doubling again this edge upon itself we obtain a right angle (§ 25). A string or a small piece of wood with two pins in it may sometimes serve as a rude substitute for the Compasses or Dividers.

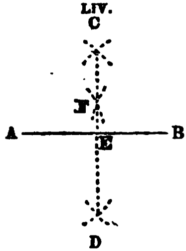
§ 135. 2. As a means of determining distance, it is often necessary to draw arcs of circles. A circle is a plane figure, bounded by a line which is everywhere equally distant from a point within called the *centre*. This line is called the *circumference*, and parts of it are called *arcs*. A line drawn from the centre to the circumference is called a *radius*. All radii, from the very definition of the circle, are equal.

3. Every problem should not only be performed, but strictly demonstrated. In the actual construction of geometrical figures, from the imperfection of our senses and instruments, absolute accuracy is of course impossible (§ 5. *g*). Still we should make this our standard, and approach as near to it as may be.

Problem I.

§ 136. *To bisect a given straight line* AB.

LIV.



From A and B as centres, with a radius greater than half of AB, describe arcs cutting each other in C and D. Draw the straight line CED. AB is bisected in E.

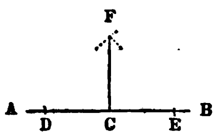
If it is not convenient to obtain two points on opposite sides of AB, they may be obtained on the same side (as C and F) by using different radii.

As the points C and D (or F) are each equally distant from A and B, the line passing through them must be a perpendicular upon the middle of AB (§ 92).

Problem II.

§ 137. *To erect a perpendicular to a straight line* AB, *at a given point* C.

LV.



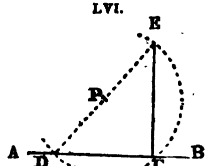
From C lay off CD and CE equal to each other. From D and E as centres, describe arcs cutting each other in F. Draw FC, which will be the perpendicular required.

As F and C are each equally distant from D and E, FC must be $\perp$ DE (§ 92).

§ 138. If the given point C is at or near the end of the line, one of the following methods may be more convenient.

SECOND METHOD.

LVI.

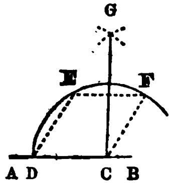


Take, as a centre, any convenient point P out of the line AB. With the radius PC draw the arc DCE, cutting AB in D. Through D and P, draw a straight line cutting this arc in E. Draw EC, which will be the perpendicular required.

Because PC = PD = PE, $\angle$ECD = 90° (§ 131).

THIRD METHOD.

LVII.



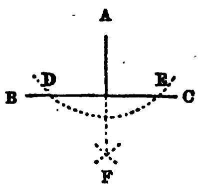
§ 139. From C as a centre, with any convenient radius CD, describe the arc DEF. In this arc take the points E and F, so that the distances DE and EF shall be = DC. From E and F as centres, with the same radius as before (or with any radius greater than half of EF), describe arcs cutting each other in G. Draw GC, which will be the perpendicular required.

If straight lines were drawn joining DE, EF, and FC, as they would be all equal to DC, the figure DEFC would be a parallelogram (§ 108). Therefore, EF ∥ AB. But, as G and C are each equally distant from E and F, GC ⊥ EF (§ 92), and therefore ⊥ AB (§ 46. *a*).

PROBLEM III.

§ 140. *From a given point* A *without a straight line* BC, *to let fall a perpendicular upon that line.*

LVIII.

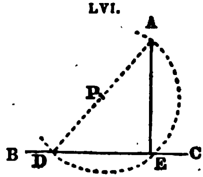


From A as a centre, with a radius greater than the distance to BC, describe an arc cutting BC in D and E. From D and E as centres, describe arcs cutting each other in F. Draw AF, which will be the perpendicular required.

As A and F are each equally distant from D and E, AF ⊥ DE (§ 92).

SECOND METHOD.

LVI.



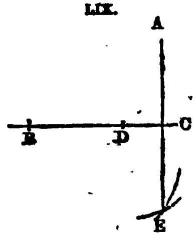
§ 141. If A is nearly opposite the point C, draw AD obliquely to BC, and bisect it in P. From P as a centre, with the radius PA, describe an arc cutting BC in E. Draw AE, which will be the perpendicular required.

Because AP = PD = PE, ∠AED = 90° (§ 131).

THIRD METHOD.

§ 142. In BC take any two points B and D; and from these as centres, with the radii BA and DA, describe arcs cutting each other in E. Draw AE, which will be the perpendicular required.

LIX.



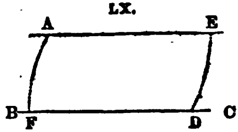
As B and D are each equally distant from A and E, therefore BC ⊥ AE (§ 92).

NOTE. In drawing perpendiculars, the instrument called a *square* is convenient.

PROBLEM IV.

§ 143. *Through a given point* A, *to draw a parallel to a given straight line* BC.

LX.



From A as a centre, with a radius greater than the distance to BC, describe the arc DE. From D as a centre, with the same radius, describe the arc AF. Take DE = AF, and draw AE, which will be the parallel required.

For, if AD be joined, as DF = AD = AE, and straight lines joining DE and AF would be equal, therefore *a*DAE = the alternate *a*ADF (§ 98). ∴ AE ∥ BC (§ 49. *c*).

NOTE. In drawing parallels, the instrument called a *parallel ruler* is convenient.

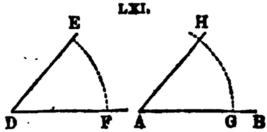
PROBLEM V.

§ 144. *At a given point* A *in a line* AB, *to make an angle equal to a given angle* D.

From D as a centre, with any radius, describe an arc cutting the sides of the angle in E and F. From A as a

centre, with the same radius, describe an arc GH. In this arc, take H at the same distance from G, that E is from F; and draw AH. HAB is the angle required.

LXI.



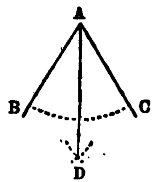
As AH = AG = DE = DF, and as straight lines joining HG and EF would be equal, *a*A = *a*D (§ 98).

Problem VI.

§ 145. *To bisect a given angle* A.

From A as a centre, with any radius, describe an arc cutting the sides of the angle in B and C. From B and C as centres, with a radius greater than half the distance between B and C, describe arcs cutting each other in D. Draw AD, which bisects the arc.

LXII.



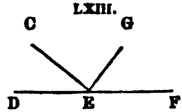
If BD and CD be joined, we have two triangles which agree in their three sides. Therefore, *a*BAD = *a*CAD (§ 98).

Problem VII.

§ 146. *Two angles of a triangle,* A *and* B, *being given, to describe the third angle.*

Take any point E in any line DEF. and make *a*DEC = *a*A, and *a*CEG = *a*B. Then, as the three angles of a triangle are equal to 180°, *a*GEF must be equal to the third angle (§ 54. *b*).

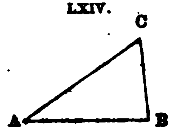
LXIII.



Problem VIII.

§ 147. *To describe a triangle, two sides and the included angle being given.*

LXIV.

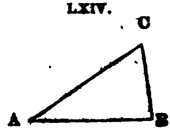


Draw AB equal to one of the given sides; make the angle BAC equal to the given angle, and AC equal to the other given side; join CB, and ACB is the triangle required (§ 40).

PROBLEM IX.

§ 148. *To describe a triangle, a side and two angles being given.*

LXIV.

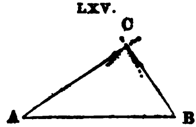


Draw AB equal to the given side. Make, at A and B, either the two given angles, or one of these and the third angle of the triangle (§ 146), as the case may require. Extend the sides, which with AB form these angles, until they meet at C. Then ACB is the triangle required (§§ 40, 90).

PROBLEM X.

§ 149. *To describe a triangle, the three sides being given.*

LXV.



Draw AB equal to one side. From A and B as centres, with two radii equal to the other two sides, each to each, describe arcs cutting each other in C. Join CA and CB, and ACB is the triangle required (§ 98).

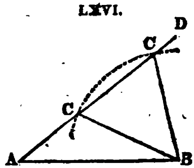
PROBLEM XI.

§ 150. *To describe a triangle, two sides and an angle opposite one of the sides being given.*

Draw AB equal to the given side adjacent to the given angle. From A draw AD making at A the given angle. From B as a centre, with a radius equal to the given side

opposite the given angle, describe an arc cutting AD in C. Join CB, and ACB is the triangle required.

LXVI.

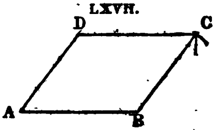


If BC is less than AB, and the angle ACB is not a right angle, this problem has a double solution, as the described arc will then cut AD in two points (§ 100).

PROBLEM XII.

§ 151. *To describe a parallelogram, an angle and the including sides being given.*

LXVII.



Draw AB equal to one of the given sides. From A draw AD equal to the other given side, and making at A an angle equal to the given angle. From D as a centre, with a radius equal to AB, and from B as a centre, with a radius equal to AD, describe arcs cutting each other in C. Join CD and CB, and ABCD is the parallelogram required.

If the opposite sides of a quadrilateral are equal, it is a parallelogram (§ 108).

REMARKS. 1. If preferred, instead of describing the arcs at C, DC may be drawn ∥ AB, and BC ∥ AD, according to Problem IV. Or DC may be drawn ∥ and = AB, and then CB be joined.

2. To draw a rectangle whose base and altitude are given, proceed as above, observing that AD is to be drawn ⊥ AB.

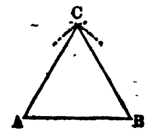
3. To draw a rhombus upon a given line and with a given angle, proceed as above, observing that the sides are to be all made equal.

PROBLEM XIII.

§ 152. *Upon a given line* AB, *to describe an equilateral triangle.*

From A and B as centres, with the radius AB, describe arcs cutting each other in C. Join CA and CB, and ACB is the triangle required.

LXVIII.

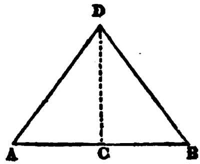


Problem XIV.

§ 153. *Upon a given line* AB, *to describe an isosceles triangle of a given altitude.*

Bisect AB in C. Draw CD ⊥ AB and equal to the given altitude. Join DA and DB, and ADB is the triangle required.

XXVI.



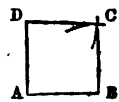
The point D, lying in a perpendicular drawn through the middle of AB, is equally distant from A and from B (§ 91).

Problem XV.

§ 154. *Upon a given line* AB, *to describe a square.*

Draw AD perpendicular and equal to AB. From D and B as centres, with a radius equal to AB, describe arcs cutting each other in C. Join CD and CB, and ABCD is the square required.

LXIX.



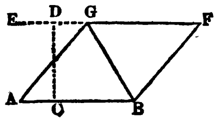
The sides are all equal and consequently parallel (§ 108); and as $aA = 90^\circ$, all the other angles must be the same (§ 109).

Problem XVI.

§ 155. *Upon a given line* AB, *to describe a triangle or a parallelogram of a given altitude and having a given angle at* A.

Draw CD ⊥ AB and = the given altitude. Through D draw EF ∥ AB. Draw AG making at A the given angle. Then if a triangle is required, join GB. If a parallelogram is required, take GF = AB and join FB (§ 108).

LXX.

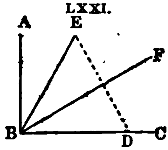


Problem XVII.

§ 156. *To trisect*[1] *the right angle* ABC.

In BC take any point D, and upon BD describe the equilateral triangle BDE (§ 152). Then bisect aEBD (§ 145), and (since aEBD $= \frac{2}{3}$ ∟, § 63. b) aABC is trisected.

LXXI.

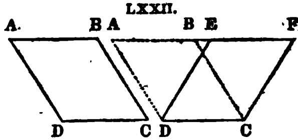


(1) Lat. *tres*, *three*, and *seco*, *to cut*, to divide into three equal parts.

PART FOURTH.

Proposition I.

LXXII.



§ 157. I. *Given*, two parallelograms, ABCD and EFCD, agreeing in base and altitude; viz. DC = DC, and the distance of AB from DC = distance of EF from DC.

Required, ABCD ≋ EFCD.

Place ABCD upon EFCD with DC upon its equal DC.
Will AB and EF lie in the same straight line? Why?

And AB ≋ EF? 108. 1.

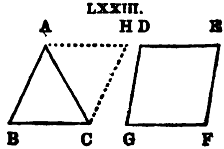
∴ AE (= AF — EF) ≋ BF (= AF — AB)? 20. *f*.

Can you now show that the triangles AED and BFC are identical? 46. *b*, 40.

But if, from the whole figure AFCD, you subtract *t*BFC, what remains? If you subtract *t*AED from the same, what remains?

∴ ABCD ≋ EFCD? 20. *f*.

LXXIII.



§ 158. II. *Given*, the triangle ABC, and the parallelogram DEFG, agreeing in base (BC = GF) and altitude.

Required, tABC ≋ DEFG.

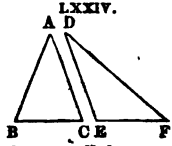
Complete the parallelogram AHCB. Then, tABC = what part of HB? 111.

But HB ≋ EG? 157.

∴ tABC = what part of EG? 20. k.

§ 159. III. *Given*, tABC and tDEF agreeing in base (BC = EF) and altitude.

LXXIV.



Required, tABC ≋ tDEF.

tABC and tDEF are each = what part of a parallelogram of the same base and altitude?

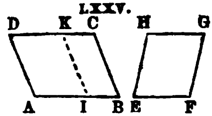
∴ tABC ≋ tDEF? 20. k.

§ 160. Theor. I. (1.) ***Two parallelograms, or (2.) two triangles, agreeing in base and altitude, are equal; and (3.) a triangle is half of a parallelogram of the same base and altitude.***

[Proved by means of superposition and identical triangles.]

§ 161. a.) *Given*, two parallelograms, ABCD and EFGH, agreeing in altitude, but AB > EF.

LXXV.



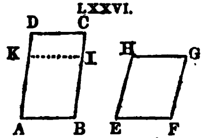
Required, ABCD ≋ EFGH.

Take AI = EF, and draw IK ∥ AD. Then, AIKD ≋ EFGH? 160. 1.

But ABCD ≋ AIKD? ∴ ABCD ≋ EFGH? 20. b.

b.) *Given*, two parallelograms, ABCD and EFGH, agreeing in base, but ABCD exceeding in altitude.

LXXVI.



Required, ABCD ≋ EFGH.

Draw IK ∥ AB, and at the same distance from AB, that GH is from EF. Then, ABIK ≋ EFGH? 160. 1.

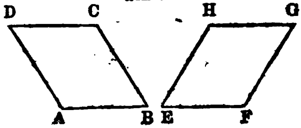
But ABCD ≋ ABIK? ∴ ABCD ≋ EFGH?

c.) As a triangle is half of a parallelogram of the same base and altitude, will the same reasoning apply to two triangles agreeing in one of these dimensions, but not in the other? Hence,

d.) COR. I. *If two parallelograms, or two triangles, agree either in base, or in altitude, that which exceeds in the other dimension is the greater of the two.*

§ 162. *e.*) *Given*, two parallelograms, ABCD and EFGH, agreeing in area and altitude.

LXXII.



Required, AB $\approx$ EF.

If AB $>$ EF, then ABCD $\approx$ EFGH? 161.

And if AB $<$ EF, then ABCD $\approx$ EFGH?

$\therefore$ AB $\approx$ EF? m. *b.*

Can you show in like manner, that, if two parallelograms agree in area and base, they must also agree in altitude?

Apply the same reasoning to two triangles.

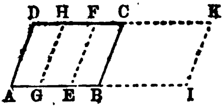
COR. II. *If two parallelograms, or two triangles, of the same area agree either in base, or in altitude, they also agree in the other dimension.*

f.) Show, in like manner, the following, which is the converse of Cor. I.:

COR. III. *If two parallelograms, or two triangles, agree either in base, or in altitude, the greater of the two exceeds in the other dimension.*

§ 163. *g.*) *Given*, a parallelogram ABCD, with a line drawn parallel to AD and including with AD a certain part or multiple of the base AB.

LXXVII.



Required, the part or multiple of ABCD which is included between this line and AD.

(1.) Let the parallel be EF, so drawn that AE $= \frac{2}{3}$AB.

Bisect AE in G, and draw GH ∥ AD. Then AG ≈ GE ≈ GB ?

∴ DG ≈ HE ≈ FB ? 160. 1.

∴ DE = what part of DB ?

If AE were any other part of AB, could it be shown, in like manner, that DE would be the same part of DB ?

(2.) Let the parallel be IK, so drawn that AI = 2AB. Then AB ≈ BI ? ∴ DB ≈ CI ?

∴ DI = what multiple of DB ?

If AI were any other multiple of AB, could it be shown, in like manner, that DI would be the same multiple of DB ?

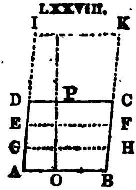
(3.) Show the converse of the above; that, if DE = $\frac{2}{3}$DB, then AE = $\frac{2}{3}$AB; and, if DI = 2DB, then AI = 2AB. See Cor. II.

Cor. IV. *A parallelogram and its base are divided or multiplied alike by lines drawn parallel to a side adjoining the base.*

If AE (Fig. LXXVII.) were $\frac{3}{8}$ of AB, what part would DE be of DB ? If AE = $\frac{17}{27}$AB, then DE = what part of DB ? If AI = 3AB, then DI = *hm*DB ? If AI = 21AB, then DI = *hm*DB ?

§ 164. *h.*) Show, in like manner, that the parallelogram ABCD, and its altitude OP, are divided or multiplied alike by EF or IK drawn parallel to AB.

LXXVIII.



Cor. V. *A parallelogram and its altitude are divided or multiplied alike by lines drawn parallel to the base.*

§ 165. *i.*) In making the comparison in §§ 163, 164, does it make any difference whether you take an *actual* part or multiple of ABCD, or a *separate* parallelogram of the same base and altitude with such a part or multiple ? 160, 18. R.

Since a triangle is half of a parallelogram of the same base and altitude, will triangles be affected in the same way as

parallelograms, by a multiplication or division of the base or altitude?

Hence the two preceding corollaries may be thus generalized:

COR. VI. (1.) *If two parallelograms, or two triangles, agree in altitude, the one is the same multiple or part of the other, which the base of the first is of the base of the second; or (2.), if they agree in base, the one is the same multiple or part of the other, which the altitude of the first is of the altitude of the second.*

How do two parallelograms of the same altitude, A and B, compare in area, if the base of A is double the base of B? if the base of A = $\frac{1}{4}$ of the base of B? = $5\frac{5}{8}$ of the base of B? = $\frac{17}{8}$ of the base of B?

How do two triangles of the same base, C and D, compare in area, if the altitude of C is half that of D? if the altitude of C = $2\frac{3}{4}$ of the altitude of D? = the altitude of D? = $99\frac{99}{100}$ of the altitude of D?

How do a parallelogram E and a triangle F of the same altitude compare in area, if the base of E is equal to the base of F? if the base of E = $\frac{1}{2}$ of the base of F? = 4 times the base of F? = $\frac{1}{3}$ of the base of F?

How do a parallelogram G and a triangle H of the same base compare in area, if the altitude of G is half that of H? double that of H? $\frac{1}{6}$ that of H? 6 times that of H?

How does a line drawn from the middle of any side of a triangle to the opposite angle divide the triangle?

---

§ 166. In any measurement, the UNIT[a] is that which is made the standard of comparison, and which, in reckoning, is counted *one*.

Thus, if you measure by feet, the linear unit, or unit of length (§ 16. *a*, *b*), is 1 foot. What is it, if you measure by yards? by rods? by miles? by inches?

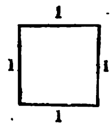
(a) Lat. unitas, *oneness, the number one.*

§ 167. With which of the regular polygons can you fill up the space about a point? 63. c.

In which of these is the base equal to the altitude? In which are the angles right angles?

What polygon, then, is the best fitted of all to be the general standard for the measurement of surface?

§ 168. In measuring *surface*, the UNIT is *the square of which one side is the linear unit.*

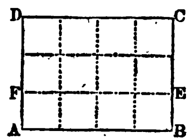


Thus, if you measure by inches, the superficial unit, or unit of surface, is a square inch. What is it, if you measure by feet? by yards? by rods? by miles?

Proposition II.

§ 169. *Given*, any rectangle ABCD.

LXXIX.



Required, the measure of its surface.

Divide AB the base, and AD the altitude, into units of length (as inches); and through the division-points draw parallels to AD and AB.

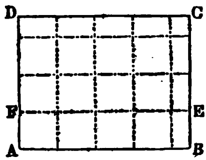
If these parallels are an inch apart, into figures of what kind is ABCD divided? Prove that these figures are all squares.

If AB is 4 inches, how many square inches does the rectangle ABEF contain?

If AD is 3 inches, how many times ABEF is the whole rectangle ABCD? 165.

How many square inches, then, does ABCD contain? How do you obtain this number?

LXXX.



If AB is 4½ inches and AD 3½ inches, how many square inches does ABEF contain, and how many times ABEF is ABCD? How many square inches, then, does ABCD contain? How do you obtain this number?

Will the same method apply, whatever may be the unit, and whatever number of units, or parts of the unit, AB or AD may contain?

In all cases, then, can you obtain the number of superficial units in a rectangle, by multiplying the number of linear units in the base, by the number of linear units in the altitude?

§ 170. THEOR. II. *The measure of a rectangle is the product of its base and altitude.*

[Proved by division into units.]

a.) When we say that *the area of a rectangle is obtained by multiplying the base into the altitude*, it is not, of course, meant that lines are converted into surfaces by the multiplication, or that one line can, properly speaking, be multiplied by another line; but simply that as many units of surface, as there are linear units in the base, are repeated as many times as there are linear units in the altitude.

§ 171. *b.*) May any side of a rectangle be taken as the base? When the base is determined, what will represent the altitude? 124. *b*, *c*.

In a square, the base = the altitude? the product of the base into the altitude = the product of the base into itself?

Hence, *the measure of a rectangle is the product of any two adjoining sides* (or it may be said to be *the product of the length into the height or width*); *and the measure of a square is the product of any side into itself.*

c.) From the mode of obtaining its measure, the rectangle of two lines (as AB and BC) is often written thus: AB ×

BC, or AB . BC. In like manner the square of a line (as AB) is often written thus: AB^2. See § 12.

d.) Since, in the comparison of magnitudes, an equal may always take the place of its equal (§ 18. *a*), a square or rectangle may be said to be the square or rectangle of any lines to which its sides are equal.

§ 172. *e.*) Divide a rectangle 5 feet long and 4 feet wide into square feet. How many rectangles do you make, each containing 5 square feet? How many do you make, each containing 4 square feet? Does it make any difference whether you divide into 4 rectangles of 5 units each, or into 5 rectangles of 4 units each? $5 \times 4 = 4 \times 5$?

In multiplying two factors, does it make any difference in the result, which you take as the multiplicand?

f.) What is the area of a rectangular board 12 ft. long and 2 ft. wide? 18 ft. long and 1½ ft. wide? 16 ft. long and 1¼ ft. wide? 12 ft. long and ¾ ft. wide?

If a rectangular field is 20 rods long and 8 rods wide, how many square rods does it contain? how many acres?

What is the area of a rectangular field 25 rods long and 20 rods wide? 16 rods long and 12½ rods wide?

If the sides, ceiling, and floor of a room are all rectangles, and the room is 20 ft. long, 16 ft. wide, and 10 ft. high, what is the area of each side? of the ceiling? of the floor? In painting the whole inside of the room, how much surface must be gone over?

What is the area of a rectangle, if the base is 10 miles and the altitude 6 miles? if the base is 12 linear units of any kind, and the altitude 7 like units?

g.) How many square inches are there in a square foot? How many square feet in a square yard? How many square yards in a square rod? How many square rods in a square furlong? How many square furlongs in a square mile?

What is the area of a square field, if one side is 20 rods? if one side is 40 rods? 80 rods?

What is the area of a square, if one side is 5 inches? if one side is 7 feet? 8 yards? 11 miles?

What is the base of a square containing 16 square feet? of one containing 36 square inches? 81 square rods?

§ 173. *h.*) How does any parallelogram compare with a rectangle of the same base and altitude? 160. 1.

What is the measure of that rectangle? 170.

What is, then, the measure of the parallelogram?

How does any triangle compare with a rectangle of the same base and altitude? 160. 3.

What is, then, its measure?

COR. I. *The measure of any parallelogram is the product, and of any triangle half the product, of the base and altitude.*

How, then, do you obtain the area of a parallelogram? of a triangle?

In obtaining the area of a triangle, does it make any difference, whether you multiply the base by the altitude and take half the product, or multiply half the base by the altitude, or multiply the base by half the altitude? Will the first, or one of the two last methods be usually found most convenient?

What is the area of a parallelogram whose base is 10 inches and altitude 7? of one whose base is 20 feet and altitude 15? of one whose base is 30 rods and altitude 9?

What is the area of a triangle whose base is 6 inches and altitude 5? of one whose base is 12 feet and altitude 8? of one whose base is 20 yards and altitude 13? of one whose base is 17 rods and altitude 16?

How many square yards are contained in a parallelogram whose base is 12 ft. and altitude 6 ft.? in one whose base is 27 ft. and altitude 4 ft.

i.) If the area of a parallelogram is 20 sq. ft. and the base 5 ft., what is the altitude?

If the area of a parallelogram is 36 sq. yds. and the altitude 4 yds., what is the base?

If the area of a parallelogram is 4 sq. yds. and the base 9 ft., what is the altitude?

If the area of a triangle is 10 sq. inches and the base 5 inches, what is the altitude?

If the area of a triangle is 35 sq. rods and the altitude 7 rods, what is the base?

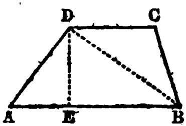
If the area of a triangle is 4 sq. yds. and the base 8 ft., what is the altitude?

If you divide an acre into 10 equal squares, what is one side of each square? If you divide it into 40 equal squares?

If a triangular piece of ground contains 5 acres, and one side is fifty rods, how far is the opposite angle from this side? 87.

In a parallelogram, if the area and the base or the altitude are given, how do you find the other dimension? In a triangle?

LXXXI.



§ 174. *k*.) In the trapezoid ABCD, *Given*, its parallel bases (§ 124. *b*) AB and DC, and its altitude DE.

Required, its area.

Divide it into two triangles by joining DB.

Then $\quad tADB = \frac{1}{2}AB \times DE$? 173.

And $\quad tDBC = \frac{1}{2}DC \times DE$?

$\therefore \quad ABCD = \frac{1}{2}(AB + DC) \times DE$?

Cor. II. *The measure of a trapezoid is half the product of the sum of its parallel bases into its altitude.*

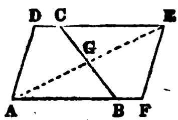
What is the area of a trapezoid, if the bases are 7 ft. and 9 ft., and the altitude 6 ft.? if the bases are 17 yds. and 13 yds., and the altitude 11 yds.?

If the area of a trapezoid is 100 sq. ft., and the two parallel sides are 13 ft. and 12 ft., what is its altitude?

If, in a trapezoidal field containing one acre, one base is 20 rods long and is 10 rods from the other base, what is the length of the other base?

§ 175. *l.*) 1. Prove Cor. II. by showing that any trapezoid (as ABCD, Fig. LXXXII.) with an identical trapezoid inverted (as CEFB) will make a parallelogram of the same altitude, but having for its base the sum of the parallel bases of the trapezoid.

LXXXII.



2. Prove that BC and AE, a diagonal of the parallelogram, bisect each other.

3. If any parallelogram (as AFED) is given, and any straight line (as BC) bisects a diagonal (as AE), prove that it also bisects the parallelogram.

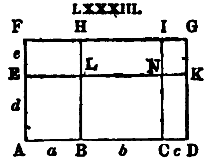
§ 176. *m.*) Can every polygon be divided into triangles? If, then, you can ascertain the base and altitude of each of these, can you obtain the area of the whole polygon? How?

§ 177. *n.*) It is often convenient to represent a line by a single Italic letter, in accordance with algebraic notation; as, a, b. The rectangle of two lines will then be expressed by two letters, either with, or more frequently without, the sign of multiplication (§ 12. 2); as, $a \times b$, or, more frequently, ab. A square will be expressed by a letter repeated, or written with the exponent ^{2}; as, aa, or a^2.

If the *base* of a parallelogram or triangle is represented by b, and its *height* by h, then the area of the parallelogram will be expressed by bh, and that of the triangle by $\frac{1}{2}bh$. If the *sum* of the parallel bases of a trapezoid is represented by s, and its *height* by h, then its area will be expressed by $\frac{1}{2}sh$.

§ 178. *o.*) 1. If the base AD of the rectangle ADGF consists of several parts AB, BC, and CD, does the whole rectangle contain the rectangles of each part of the base into the altitude AF?

LXXXIII.



$$ADGF = AB \times AF + BC \times AF + CD \times AF?$$

Point out each of these rectangles upon the figure (in which BH and CI are drawn ∥ AF). See § 171. *d*.

2. If the altitude AF consists of several parts AE and EF, does the whole rectangle contain the rectangles of the base into each part of the altitude?

$$ADGF = AD \times AE + AD \times EF?$$

Point out each of these rectangles upon the figure (in which EK is drawn ∥ AD).

3. If both the base and the altitude are divided (as in the figure above), does the whole rectangle contain the rectangles of each part of the base into each part of the altitude?

$$ADGF = AB \times AE + BC \times AE, \&c.$$

(complete the equation, and point out each rectangle upon the figure).

COR. III. *The rectangle of two lines is the sum of the rectangles of each part of the one into each part of the other.*

How, then, do you find the area of a rectangle whose sides consist of several parts?

(a) If the parts of the base, in Fig. LXXXIII., are represented by a, b, and c, and the parts of the altitude by d and e (§ 177), then

$$(a+b+c) \times (d+e) = ad+bd+cd+ae+be+ce.$$

Point out each term of this equation upon the figure.

(b) Let $a = 4$ ft., $b = 6$ ft., $c = 2$ ft., $d = 5$ ft., and $e = 3$ ft. What is then the whole base? the whole altitude? What is each of the rectangles ad, bd, &c.? What is their sum? Is this the same with the area of the whole rectangle as obtained by directly multiplying the whole base into the whole altitude?

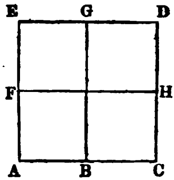
§ 179. *p*.) If the base of a rectangle is divided into 4 parts, and the altitude into 3 parts, into how many parts will the rectangle be divided by parallels drawn through the division-points? Into how many parts will it be divided, if the base is divided into 5 parts and the altitude into 6 parts? if both

the base and the altitude are each divided into 10 parts? into 12?

If all the parts both of the base and of the altitude are equal, into figures of what kind will the rectangle be divided?

If the sides of a square are each divided into two equal parts, into how many smaller squares will it be divided by parallels drawn through the division-points? If they are each divided into 3 equal parts? into 5? into 8? into 20?

LXXXIV.



What part of the square of a line is the square of its half? of its third? of its fourth?

If R = 5S, then $R^2 = hmS^2$? $R^2 = hmS^2$, if R = 7S? if R = 10S? if R = 40S?

Cor. iv. *The square of a line is equal to four times the square of its half, to nine times the square of its third*, &c.; and, in general, *whatever number expresses the relation of one line to another, the square of this number expresses the relation of the square of the first line to the square of the second.*

(a) If a line be represented by a, then

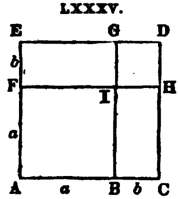
$$(2a)^2 = 4a^2,\ (\tfrac{1}{2}a)^2 = \tfrac{1}{4}a^2,\ (5a)^2 = 25a^2,\ \&c.$$

(b) Twice the square of the half of a line = what part of the square of the whole line?

(c) How does the square of a line (as AC^2, Fig. LXXXIV.) compare with the rectangle of the whole line into its half (as AC × AB)? How does this rectangle compare with the square of the half (as AB^2)?

§ 180. *p.*) *Given*, AC = AB + BC.

LXXXV.



Required, the square of AC in terms of AB and BC (§ 22).

(AB + BC) × (AB + BC) = AB × AB + BC × AB + AB × BC + BC × BC?

Which of these terms[b] may be more briefly written? 12. 2.
Which two, as equal, may be united? 172. e.

∴ $(AB+BC)^2$, or AC^2, $=$ $AB^2+2AB\times BC+BC^2$?

Show, in Fig. LXXXV. (where $AF=AB$, $BG \parallel AE$, and $FH \parallel AC$), that AD, the square of AC, contains each of these smaller squares and rectangles.

COR. V. *The square of the sum of two lines is equal to the sum of their squares,* PLUS *twice their rectangle.*

(a) If the two lines are represented by a and b, then

$$(a+b)^2 = a^2 + 2ab + b^2.$$

(b) Let $a = 3$ inches, and $b = 2$ inches. What is then the area of a^2, of $2ab$, and of b^2? What is the sum of these areas? Is this the same with the area of the whole square as obtained by a single multiplication?

(c) Find according to this method the area of a square whose base is 4 ft. + 3 ft.; of one whose base is 7 ft. + 5 ft.; of one whose base is 10 ft. + 5 ft.; of one whose base is 10 ft. + 9 ft.; of one whose base is 100 ft. + 12 ft.

(d) If $a = b$, show that the equation above, $(a+b)^2 = a^2 + 2ab + b^2$, may be changed into $(2a)^2 = 4a^2$; and compare Cor. IV. (§ 179).

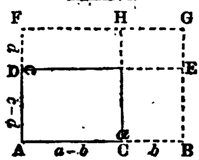
§ 181. *r.*) Cor. V. might be thus extended to any number of lines: *The square of the sum of any number of lines is equal to the sum of their squares, plus twice the rectangle of each pair which can be formed by combining them.* Thus,

$$(a+b+c)^2 = a^2 + 2ab + b^2 + 2ac + 2bc + c^2.$$

Draw the square of a line consisting of three parts, and so divide it by parallels as to show that it contains each of these smaller squares and rectangles.

(b) Gr. τέρμα, *bound, limit;* Lat. terminus, Fr. terme. The parts of a quantity which are connected by the signs + and − are called *terms*. For another application of the word, see § 22.

LXXXVI.



§ 182. *s.*) A line may not only be regarded as the sum, but also as the difference, of other lines; thus,

$$AC = AB - CB.$$

From the rectangle of AB into AD, what must you subtract, to obtain the rectangle of AC into AD?

$\therefore$ $AC \times AD$, or $(AB - CB) \times AD$, $\approx$ $AB \times AD - CB \times AD$?

But AD may itself be regarded as the difference between AF and DF; and then the rectangle of AC and AD will be thus represented: $(AB - CB) \times (AF - DF)$.

From the rectangle of AC into AF, what must you subtract to obtain the rectangle of AC into AD?

$\therefore$ $\quad AC \times AD \approx AC \times AF - AC \times DF$?

But $\quad AC \times AF \approx AB \times AF - CB \times AF$?

And $\quad AC \times DF \approx AB \times DF - CB \times DF$?

Must, then, this second quantity $(AB \times DF - CB \times DF)$ be subtracted from the first $(AB \times AF - CB \times AF)$, in order to obtain the value of $AC \times AD$?

In subtracting $AB \times DF$, do you subtract too much or too little?

How much do you subtract more than you should?

What, then, must you add, to make the result right?

$\therefore$ $AC \times AD$, or $(AB - CB) \times (AF - DF)$, $\approx$ $AB \times AF - CB \times AF - AB \times DF + CB \times DF$?

Point out each of these rectangles upon the figure; and show that the rectangle HE is subtracted twice, and therefore must be added, to make the result right.

NOTE. Quantities to which the sign — is prefixed are termed *negative*[c] as diminishing the amount; those to which the sign + is prefixed (which is also to be understood, when a quantity, standing alone or beginning a series, has no sign

(c) Lat. negativus, from nego, *to deny*, as denying value.

prefixed) are termed *positive*[d] as increasing the amount. It will be observed that when a positive and a negative quantity are multiplied together, the result is negative; and that when two positive quantities, or two negative quantities, are multiplied together, the result is positive. Hence has arisen what is termed *the law of the signs in multiplication;* viz. LIKE SIGNS (+ +, or — —) produce *plus* (+); UNLIKE (+ —, or — +), *minus* (—).

COR. VI. *The difference of two lines is multiplied in the same manner as their sum, observing the law of the signs in multiplication.*

(a) If AB, in Fig. LXXXVI., is represented by a, CB by b, AF by c, and DF by d, then AC $= a - b$, and AD $= c - d$; and we have, as an expression of the value of AC × AD,

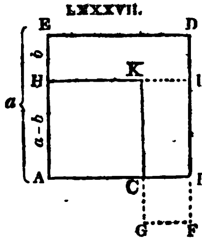
$$(a-b)\times(c-d) = ac - bc - ad + bd.$$

Point out each term of this equation upon the figure.

(b) Let $a = 10$ ft., $b = 4$ ft., $c = 8$ ft., and $d = 3$ ft., what is then the value of each of the terms obtained by the multiplication? What is the whole amount? Is this the same with the product obtained by directly multiplying the value of $a - b$ into the value of $c - d$?

§ 183. *t.*) *Given*, AC = AB — CB.

LXXXVII.



Required, the square of AC in terms of AB and CB.

(AB — CB) × (AB — CB) = AB × AB — CB × AB — AB × CB + CB × CB?

∴ $(AB - CB)^2$, or AC^2, = $AB^2 - 2AB \times CB + CB^2$?

Show, in Fig. LXXXVII. (where AH = AC, HI ∥ AB, and CK ∥ AH), that, if from AD (the square of AB) + CF (the square of CB) you take twice the rectangle of AB and CB, there remains the square of AC.

(d) L. positivus, from pono, *to place*, as placing or giving value.

COR. VII. *The square of the difference of two lines is equal to the sum of their squares,* MINUS *twice their rectangle.*

(a) If the two lines are represented by a and b, then

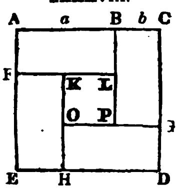
$$(a - b)^2, \text{ or } (b - a)^2, = a^2 - 2ab + b^2.$$

(b) Let $a = 5$ ft. and $b = 2$ ft. What is then the value of a^2, of $-2ab$, and of b^2? Is the amount the same with that which is obtained by directly multiplying the value of $a - b$ [$5 - 2 = 3$] into itself?

(c) Find, according to this method, the area of a square whose base is $10 - 1$ ft.; of one whose base is $50 - 3$ ft.; of one whose base is $100 - 7$ ft.

§ 184. *u.*) If the square of the *sum* of two lines is *greater* than the sum of their squares by twice their rectangle (§ 180), and the square of their *difference* is *less* than the sum of their squares by twice their rectangle (§ 183), how much does the square of their sum exceed the square of their difference?

LXXXVIII.



Show, in Fig. LXXXVIII. (where AB = CG = DH = EF, and BP, GO, HK, and FL are drawn parallel to the sides of ACDE, the square of AC the sum of AB and BC), that KLPO is the square of the difference of AB and BC, and that AD, the square of the sum, exceeds KP, the square of the difference, by 4 times the rectangle of AB and BC.

COR. VIII. *The square of the sum of two lines exceeds the square of their difference by four times their rectangle.*

(a) If the greater line is represented by a, and the less by b, then

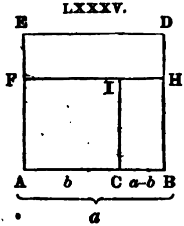
$$(a + b)^2 - (a - b)^2 = 4ab.$$

(b) Let $a = 5$ ft. and $b = 3$ ft. What is then their sum? their difference? the square of their sum? the square of their difference? Show that the square of the sum exceeds the

square of the difference by 4 times the rectangle of the two lines.

Let $a = 8$ ft. and $b = 4$ ft.; and then show the same.

LXXXV.



§ 185. *v.*) *Given*, two lines AB and AC.

Required, the rectangle of their sum and difference in terms of their squares.

$(AB + AC) \times (AB - AC) = AB^2 + AC \times AB - AB \times AC - AC^2$?

Is the value at all affected by adding $AC \times AB$, and subtracting $AB \times AC$? 21. *a.*

May we then cancel $+ AC \times AB - AB \times AC$?

$\therefore \quad (AB + AC) \times (AB - AC) = AB^2 - AC^2$?

Show, in Fig. LXXXV., that if from AD, the square of AB, you take AI, the square of AC, there remain the rectangles of each line into their difference.

Are these together equal to the rectangle of the sum of the two lines into their difference? 178.

COR. IX. *The rectangle of the sum and difference of two lines is equal to the difference of their squares.*

(a) If the greater line is represented by a, and the less by b, then

$$(a + b) \times (a - b) = a^2 - b^2.$$

(b) What, according to this Corollary, is the difference between the square of 5 ft. and the square of 3 ft.? Do you obtain the same result by subtracting the square of 3 ft. from the square of 5 ft.?

(c) What must you add to the square of 5 ft. to obtain the square of 7 ft.? to the square of 10 ft. to obtain the square of 15 ft.? to the square of 45 ft. to obtain the square of 55 ft.?

(d) What must you subtract from the square of 8 ft. to obtain the square of 6 ft.? from the square of 20 ft. to obtain the square of 17 ft.?

(e) What must you add to the square of 10 ft. to obtain the square of 11 ft.? to the square of 11 ft. to obtain that of 12 ft.? to the square of 12 ft. to obtain that of 13 ft.? to the square of 100 ft. to obtain that of 101 ft.?

What must you add to the square of any number to obtain the square of the next higher number?

(f) What must you subtract from the square of 10 ft. to obtain that of 9 ft.? from the square of 9 ft. to obtain that of 8 ft.? from the square of 100 ft. to obtain that of 99 ft.?

What must you subtract from the square of any number to obtain the square of the next lower number?

(g) As any two numbers may be resolved into the sum and difference of two numbers, this Corollary suggests a method of finding the product of two numbers which is sometimes convenient.

Find, by applying this Corollary, the area of a parallelogram whose base is 22 (= 20 + 2) ft. and altitude 18 (= 20 — 2) ft.

What is the area of a rectangular field 53 rods long and 47 rods wide? of a rectangular field 97 rods long and 83 rods wide?

§ 186. *w.*) In Fig. LXXXV. (§ 185), the rectangle $AB \times AC = AC^2 +$ what rectangle?

And $AB \times AC = AB^2 -$ what rectangle?

COR. X. *The rectangle of two lines is equal to the square of the less plus the rectangle of the less and the difference of the two; or to the square of the greater minus the rectangle of the greater and the difference of the two.*

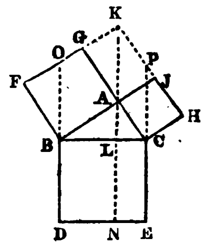
(a) Compare the rectangle of 5 ft. and 7 ft. with the square of 5 ft., and with the square of 7 ft.

Compare the rectangle of 10 ft. and 20 ft. with the squares of 10 ft. and 20 ft.

(b) Apply this Corollary in proof of Cor. IX.

PROPOSITION III.

LXXXIX.



§ 187. *Given*, a triangle ABC, right angled at A.

Required, $BC^2 \approx AB^2 + AC^2$.

Let BE be the square of BC, AF of AB, and AH of AC. Through A, draw KLN ∥ BD; produce FG and HJ till they meet KA; and produce DB to O, and EC to P.

*a*FBA ≈ *a*OBC? 27.

∴ Subtracting OBA, *a*FBO ≈ *a*ABC? 20.*f*.

In what three parts, then, do *t*FBO and *t*ABC agree?

∴ OB ≈ BC ≈ BD? 40, 124.

∴ Parallelogram BN ≈ BK? 160. 1.

But, as FK ∥ BA, BK ≈ AF?

∴ BN ≈ AF?

Show, in like manner, PC ≈ BC ≈ CE; and then, CN ≈ CK ≈ AH.

But, BE ≈ BN + CN? ∴ BE ≈ AF + AH?

§ 188. THEOR. III. *In a right triangle, the square of the hypotenuse is equal to the sum of the squares of the other two sides.*

[Proved by dividing the square of the hypotenuse into two rectangles, each of which, by the aid of Theor. I., is shown to be equal to one of the other squares.]

§ 189. *a*.) If the base of a right triangle is 8 ft. and the perpendicular 6 ft., what is the square of the base? the square of the perpendicular? the square of the hypotenuse? What, then, is the hypotenuse?

What is the hypotenuse of a right triangle, if the base is 3 ft. and the perpendicular 4 ft.? if the base is 12 yds. and

the perpendicular 5 yds.? if the base is 40 rods and the perpendicular 30 rods?

In any right triangle, if the base and perpendicular are given, how do you find the hypotenuse?

§ 190. *b.*) $AB^2 = BC^2 - AC^2$? And $AC^2 = BC^2 - AB^2$?

COR. I. *In a right triangle, the square of one of the sides containing the right angle is equal to the difference of the squares of the other two sides.*

c.) If the base of a right triangle is 4 ft. and the hypotenuse 5 ft., what is the square of the perpendicular? what is the perpendicular?

What is the base of a right triangle, if the square of the hypotenuse is 100 sq. yds. and the square of the perpendicular 36 sq. yds.? if the hypotenuse is 15 ft. and the perpendicular 9 ft.? if the hypotenuse is 50 rods and the perpendicular 30 rods?

In any right triangle, if the hypotenuse and one of the other sides are given, how do you find the third side?

d.) In Fig. LXXXIX., how does AC^2 ($= BC^2 - AB^2$) compare with the rectangle of the sum and difference of BC and AB? 185.

Apply § 185 to finding the perpendicular of a right triangle whose hypotenuse is 25 ft. and base 24 ft.; of one whose hypotenuse is 20 ft. and base 16 ft.

§ 191. *e.*) In any rectangle, how does the square of a diagonal compare with the sum of the squares of any two adjoining sides?

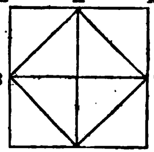
What is the length of a straight road between opposite corners of a rectangular township 8 miles long and 6 miles wide?

If a rectangular field is 16 rods long and 12 rods wide, how much distance do you save, in going from one corner to the opposite corner, by walking straight across the field, instead of walking upon the two sides?

How do the squares of the four sides of a rectangle together compare with the squares of its two diagonals?

§ 192. *f*.) If $aBAC = 90°$, and $AB = AC$, then $BC^2 = hmAB^2$?

LXXXIV.



Draw CD ∥ AB, and BD ∥ AC. Join AD; and through the points A, D, B, and C, draw EH and FG ∥ BC, and EF and HG ∥ AD. Point out in the figure the squares of AB and BC, and show from the figure that $BC^2 = 2AB^2$.

$$AB^2 = hmAE^2\ ?$$

What part, then, is any square of the square of its diagonal?

How many identical triangles does Fig. LXXXIV. contain? What squares in the figure contain two of these triangles? four? eight?

Cor. II. *Any square is half the square of its diagonal.*

g.) If one side of a square is 5 ft., what is the square of its diagonal? If one side is 8 ft.? If one side is 10 ft.?

What is the base of a square, if the square of its diagonal is 72 sq. yds.? if the square of its diagonal is 98 sq. rods? if the square of its diagonal is 242 sq. rods.

What is the distance between the opposite corners of a square field containing 50 sq. rods? of one containing 1 acre and 40 sq. rods? of one containing $\frac{4}{5}$ of an acre?

§ 193. *h*.) In Fig. LXXXIX., $AB^2 \approx BC \times BL$? 187.

And $AC^2 \approx BC \times LC$?

Again $AB^2 + AC^2 \approx BL^2 + LC^2 + 2AL^2$? 188.

And $BC^2 \approx BL^2 + LC^2 + 2BL \times LC$? 180.

∴ $AL^2 \approx BL \times LC$? 20. *f*, *k*.

Cor. III. *If the hypotenuse of a right triangle be divided into two segments*[e] *by a perpendicular drawn from the right angle,* (1.) *the square of either of the other sides is equal to the rectangle of the hypotenuse into its segment adjoining*

(e) Lat. segmentum, *portion, division, part cut off,* from seco, *to cut.*

that side; and (2.) *the square of this perpendicular is equal to the rectangle of the two segments.*

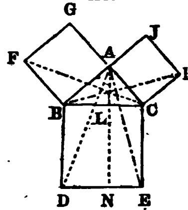
i.) If $BL = \frac{1}{2}BC$, AB^2 = what part of BC^2? $AB^2 \approx AC^2$?

If $BL = \frac{1}{3}BC$, AB^2 = what part of BC^2? of AC^2? If $BL = \frac{1}{4}BC$?

Whatever part BL is of BC, is AB^2 the same part of BC^2?

Show from Cor. III. 2., that, if $BL = LC$, then $AL = BL$. Compare § 131.

XC.

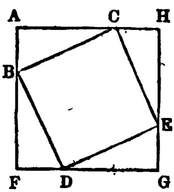


§ 194. *k.*) The following is *Euclid's Method* of demonstrating Theor. III.

Draw AN ∥ BD; and join AD, FC, AE, and HB. Show that *t*ABD and *t*FBC are identical; and that, therefore (since $tABD = \frac{1}{2}BN$, and $tFBC = \frac{1}{2}AF$), BN is = AF. Show, in like manner, that *t*ACE and *t*HCB are identical; and, consequently, CN = AH. ∴ BE = AF + AH.

§ 195. *l.*) *Third Method.* Join OP, in Fig. LXXXIX., and show that BOPC is the square of BC. Show that the triangles BFO, OKP, PHC, BAC, AGK, and KJA are identical; and that, if you take the first three from the pentagon BFKHC, BC^2 remains, while, if you take from it the last three, AB^2 and AC^2 remain. ∴ $BC^2 = AB^2 + AC^2$. 20.*f.*

LXXXVIII.

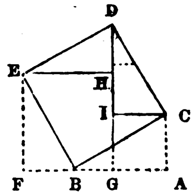


§ 196. *m.*) *Fourth Method.* Upon the hypotenuse BC, describe the square BDEC; and through D and E, draw FG and GH parallel to AC and AB, and forming, with these lines produced, the quadrilateral AFGH. Show that the triangles ABC, FDB, GED, and HCE are identical, and are together = $2AB \times AC$; so that AFGH is = $BC^2 + 2AB \times AC$. Show that AFGH is the square of AB +

AC, and is, therefore, $= AB^2 + AC^2 + 2AB \times AC$.
$\therefore BC^2 = AB^2 + AC^2$. 29. *f*.

§ 197. *n.*) *Fifth Method.* Divide BCDE, the square of the hypotenuse BC, as in Fig. XCI. (where the angles DEH, CDI, and BCI are each $=$ CBA). Put tIDC into the position ABC, and tHED into the position FEB; and show that you have thus made, from the square of BC, the two squares of AC and AB.

XCI.



Illustrate this method by the actual dissection of a square drawn upon paper.

§ 198. *o.*) This Proposition, which has been named, from its discoverer, the Pythagorean, is the most celebrated in Geometry; and there is none which has more exercised the ingenuity of mathematicians in devising different methods of proof. Hoffmann, in his Treatise upon the Proposition (Mayence, 1819), exhibits no fewer than thirty-two methods differing more or less from each other. Of these, the methods given above are the most celebrated and beautiful.

§ 199. *p.*) $(a + b)^2 = a^2 + 2ab + b^2$. 180. a.
And $(a - b)^2 = a^2 - 2ab + b^2$. 183. a.
$\therefore$ Adding the equations, and cancelling $+ 2ab - 2ab$,

$$(a + b)^2 + (a - b)^2 = 2a^2 + 2b^2 .$$

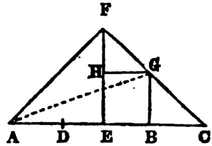
Dividing by 2, $\frac{1}{2}(a + b)^2 + \frac{1}{2}(a - b)^2 = a^2 + b^2 .$

Cor. IV. (1.) *The squares of two lines are together equal to half the squares of their sum and difference.* (2.) *The squares of the sum and difference of two lines are together equal to twice the squares of the lines.*

Apply this Corollary to two lines, one of which is 6 ft. and the other 4 ft. Apply it to two lines, one of which is 9 ft. and the other 3 ft.

§ 200. *q.*) Each part of Cor. IV. may be proved by a separate and beautiful application of Theor. III.

XCII.



(1.) Let the two lines be AB and BC, joined at B so as to form one straight line. Then their sum will be AC, and their difference DB, obtained by taking, from AB, AD = BC.

Bisect AC in E. Is DB also bisected in E? Why?

At E raise EF $\perp$ and = AE; and join FA and FC. Draw BG $\parallel$ EF, meeting FC at G; and draw GH $\parallel$ AB.

What lines in the figure are each equal to AE? to BC? to EB?

What right-angled isosceles triangles are there in the figure?

Join AG. What kind of triangles are AFG and ABG?

$\therefore$ $AB^2 + BG^2$ = square of what line?

And AG^2 = squares of what two other lines?

$\therefore$ $AB^2 + BC^2\ (= BG^2) \approx AF^2 + FG^2$?

But AF^2 = twice the square of what line? 192. *f.*

And FG^2 = twice the square of what line?

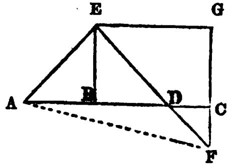
$\therefore$ $AB^2 + BC^2 \approx 2AE^2 + 2EB^2$?

But $2AE^2$ = half of what square? 179. b.

And $2EB^2$ = half of what square?

$\therefore$ $AB^2 + BC^2 \approx \frac{1}{2}AC^2 + \frac{1}{2}DB^2$?

XCIII.



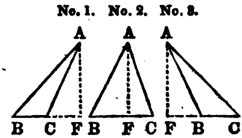
(2.) Let the two lines be AB and BC, joined at B so as to form one straight line. Then their sum will be AC, and their difference DC, obtained by taking, from BC, BD = AB.

At B raise BE $\perp$ and = AB. Join AE, and through D draw EDF meeting at F with the line GCF drawn $\perp$ AC through *p*C. Draw EG $\parallel$ BC, and join AF.

Show, by a method of proof similar to the preceding, that $AC^2 + DC^2\ (= CF^2)$ are $= AF^2$; and, therefore, $= AE^2 + EF^2$; and, therefore, $= 2AB^2 + 2BC^2$.

PROPOSITION IV.

XCIV.



§ 201. *Given*, any side of a triangle AB, opposite to an oblique angle C.

Required, the square of AB in terms of the other sides.

From aA, either of the angles adjacent to AB, draw AF $\perp$ the opposite side BC, produced if necessary. Three cases will be observed. (1.) If aC is obtuse, AF falls without the triangle on the side of C. (2.) If aC and aB are both acute, AF falls within the triangle. (3.) If aB is obtuse, AF falls without the triangle on the side of AB. The mode of investigating the three cases is the same, and the result the same except in the sign of one of the terms.

As aAFB $= 90°$, $AB^2 \approx AF^2 + BF^2$? 188.

But, in Case 1, as $BF = CF + BC$,

$BF^2 \approx CF^2 + BC^2 + 2BC \times CF$? 180.

$\therefore$ $AB^2 \approx AF^2 + CF^2 + BC^2 + 2BC \times CF$? 18. *a.*

But $AF^2 + CF^2 =$ square of what line? 188.

$\therefore$ $AB^2 \approx AC^2 + BC^2 + 2BC \times CF$?

In Cases 2 and 3, as $BF = BC - CF$, or $CF - BC$,

$BF^2 \approx CF^2 + BC^2 - 2BC \times CF$? 183.

$\therefore$ $AB^2 \approx AF^2 + CF^2 + BC^2 - 2BC \times CF$?

$\therefore$ $AB^2 \approx AC^2 + BC^2 - 2BC \times CF$?

How does the value of AB^2 in Cases 2 and 3 differ from its value in Case 1?

Universally, $AB^2 = AC^2 + BC^2 +$ or $- 2BC \times CF$.

Trace upon the figure, and write out separately, the demonstration of each case.

§ 202. THEOR. IV. *The square of any side of a triangle subtending an* OBTUSE *angle is* GREATER, *and of any side subtending an* ACUTE *angle* LESS, *than the sum of the squares of the other two sides, by twice the rectangle of either of these sides into the distance between a perpendicular let fall upon it from the opposite vertex and the given angle.*

[Proved by means of Theor. III., and §§ 180, 183.]

§ 203. *a.*) 1. *Given*, in *t*ABC (Fig. XCIV.), $AB^2 > AC^2 + BC^2$.

Required, $aC \approx 90^\circ$.

If $aC = 90^\circ$, then $AB^2 \approx AC^2 + BC^2$? 188.

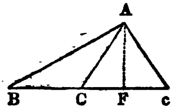
And, if $aC < 90^\circ$, then $AB^2 \approx AC^2 + BC^2$?

$\therefore$ $aC \approx 90^\circ$? 23. *b.*

2. Show by the same mode of reasoning, that, if $AB^2 < AC^2 + BC^2$, then $aC < 90^\circ$; and, if $AB^2 = AC^2 + BC^2$, then $aC = 90^\circ$. Hence (uniting §§ 188, 202, 203. *a*),

b.) COR. I. *The square of any side of a triangle is* $=$, $>$, *or* $<$ *the sum of the squares of the other two sides, according as the angle which it subtends is* $=$, $>$, *or* $<$ *a right angle; and the converse.*

XL.



c.) Show, from § 78, that the sides containing the right angle AFB are greater than those containing the obtuse angle ACB, and less than those containing the acute angle AcB.

§ 204. *d.*) If the formula above ($AB^2 = AC^2 + BC^2 +$ or $- 2BC \times CF$) were applied to the hypotenuse of a right triangle, as the perpendicular AF must now fall upon AC, how much would CF be? How much, then, would $+$ or $- 2BC \times CF$ be? Do you obtain, then, the same result as in § 187? This formula, therefore, may be applied to any side of any triangle.

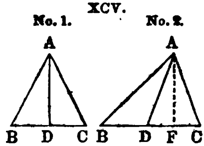
§ 205. *e.*) As, in Fig. XCIV., $AB^2 = AF^2 + BF^2$, and $AC^2 = AF^2 + CF^2$,

$AB^2 - AC^2 = BF^2 - CF^2$? 20. *f.*

COR. II. *The difference between the squares of any two sides of a triangle is the same with the difference between the squares of their distances from a perpendicular let fall upon the third side from the opposite angle.*

PROPOSITION V.

XCV.



§ 206. *Given,* any triangle ABC, with a line drawn from the middle of any side BC to the opposite angle.

Required, the sum of the squares of AB and AC, in terms of BC and AD.

If AD is $\perp$ BC, as in No. 1, then $AB^2 =$ what two squares? 198.

And $AC^2 =$ what two squares?

$\therefore$ $AB^2 + AC^2 =$ what four squares?

If AD is not $\perp$ BC, as in No. 2, draw AF $\perp$ BC.

Then $AB^2 =$ what two squares + what rectangle? 202.

And $AC^2 =$ what two squares — what rectangle?

In adding these equations, what terms cancel each other? 21. *a.*

$\therefore$ $AB^2 + AC^2 =$ what four squares?

But, in the four squares, in both cases, what square is taken twice?

And $BD^2 + DC^2 = 2BD^2$?

$\therefore$ $AB^2 + AC^2 = 2BD^2 + 2AD^2$?

§ 207. THEOR. V. *The squares of any two sides of a triangle are together equal to twice the squares*

of half the third side, and of the straight line drawn from the middle of this side to the opposite angle.

[Proved by applying Theorems III. and IV.]

§ 208. *a.*) What kind of a triangle is ABC in Fig. XCV., No. 1? 91, 206.

If AB and AC are each = 5 ft. and BC = 6 ft., what is the length of AD? What is the area of the triangle? 178.

If BC = 12 ft. and AD = 8 ft., what is the length of the equal sides AB and AC?

There is a field in the shape of an isosceles triangle, of which the area is $7\frac{1}{2}$ acres and the altitude 40 rods; what is the length of each side?

If the three sides of an isosceles triangle are given, how do you find the altitude? the area?

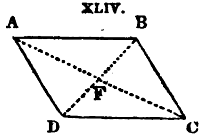
b.) If, in No. 2, AB = 9 ft., AC = 7 ft., and AD^2 = 29 ft., what is the length of BC?

c.) If *a*BAC = 90°, then $2BD^2 + 2AD^2$ = what single square? 131, 179.

§ 209. *d.*) *Given*, any parallelogram ABCD.

Required, the sum of the squares of its sides, in terms of the diagonals.

XLIV.



As, in *t*ABC, BF bisects AC, $AB^2 + BC^2$ = what? 207.

So, in *t*CDA, $CD^2 + DA^2$ = what?

∴ Adding the equations, $AB^2 + BC^2 + CD^2 + DA^2$ ≋ $4AF^2 + 2BF^2 + 2FD^2$?

But $4AF^2$ = the square of what line? 179.

And $2BF^2 + 2FD^2$ = what single square?

∴ $AB^2 + BC^2 + CD^2 + DA^2$ ≋ $AC^2 + BD^2$?

COR. *In any parallelogram, the squares of the sides are together equal to the squares of the diagonals.*

If AB = 5 ft., BC = 5 ft., and BD = 6 ft., what is the length of AC?

If AD = 6 ft., DC = 8 ft., and BD^2 = 79 sq. ft., what is the length of AC?

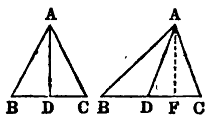
PART FIFTH.

PROBLEMS.

Problem I.

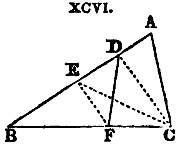
§ 210. *To bisect a given triangle* ABC.

XCV.



1. *By a line drawn from one angle* A. Bisect BC in D, and join AD, which will bisect the triangle (§ 160).

XCVI.



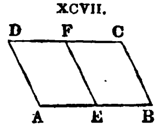
2. *By a line drawn from a point* D *upon one side* AB. Join DC. Bisect AB in E, draw EF ∥ DC, and join DF, which will bisect *t*ABC.

Join EC. Then *t*AEC = ½*t*ABC. But *t*DFC = *t*DEC (§ 160). ∴ ADFC = ½*t*ABC.

Problem II.

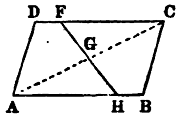
§ 211. *To bisect a given parallelogram* ABCD.

XCVII.



1. *By a line drawn parallel to one side* AD. Bisect AB, an adjoining side, in E; and draw EF ∥ AD. EF bisects ABCD (§ 163).

LXXXII.



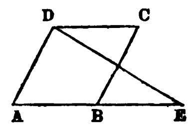
2. *By a line drawn from one angle* A. Join AC, and the parallelogram is bisected (§ 110).

3. *By a line drawn through any other point* F, *either without or within* ABCD. Draw a diagonal AC, and bisect it in G (which may be done by joining DB). Through F and G draw FGH, which will bisect ABCD (§ 175. 3).

Problem III.

§ 212. *To describe a triangle equal to a given polygon of any greater number of sides.*

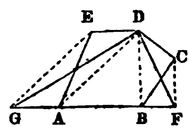
XCVIII.



1. If the given polygon is a parallelogram, as ABCD, produce the base AB to E, making BE = AB, and join DE. ADE is the triangle required.

ABCD and *t*ADE are each = ½ of a parallelogram of the same altitude upon the base AE (§§ 160, 165).

XCIX.



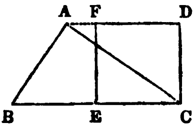
2. If the given polygon is not a parallelogram, as ABCDE, reduce the number of sides to three as follows. Join DB, and from C draw CF ∥ DB, to meet AB produced. Join DF; and since *t*DFB = *t*DCB (§ 160), the quadrilateral AFDE is equal to the pentagon ABCDE. By drawing DG in like manner, reduce AFDE to the equal triangle GFD. Whatever may be the number of sides in the given polygon, it may thus, by successive steps, be reduced to a triangle.

Note. Any polygon in which there are given a sufficient number of sides and angles or diagonals to determine it, may be constructed by drawing successively its several parts, according to the same general rules as those which apply to the construction of triangles.

PROBLEM IV.

§ 213. *To describe a rectangle equal to a given triangle* ABC.

C.

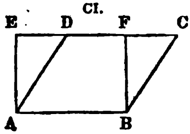


Through A, draw AD ∥ BC. Bisect BC in E; draw EF and CD ⊥ BC; and ECDF will be the rectangle required (§ 173).

PROBLEM V.

§ 214. *To describe a rectangle equal to a given oblique parallelogram* ABCD.

CI.



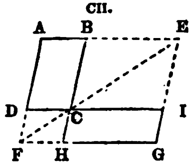
Draw AE and BF ⊥ AB, and meeting CD produced as far as may be necessary. ABFE is the rectangle required (§ 160).

NOTE. Instead of the rectangle, both in this problem and in the preceding, a parallelogram having any required angle may be described, by making this angle at the base, in the place of one of the right angles.

PROBLEM VI.

§ 215. *To describe a parallelogram which shall be equal and equiangular to a given parallelogram* ABCD, *and which shall have one side of a given length.*

CII.



Produce one side of the given parallelogram AB, making BE equal to the given length; and draw ECF meeting AD produced in F. Draw EG and FG parallel to AD and AB, and produce BC and DC to H and I. CIGH is the parallelogram required.

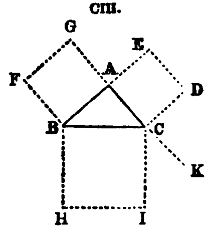
tEAF = tFGE, tEBC = tCIE, and tCDF = tFHC (§ 110). ∴ Subtracting equals from equals, ABCD = CIGH.

§ 216. Note. When a parallelogram is divided, as AFGE above, into four smaller parallelograms by lines drawn through a point in a diagonal, the two through which the diagonal passes are termed the *parallelograms about the diagonal*, and the other two are termed their *complements*[f]. It appears from the preceding section, that *these complements are always equal.*

Problem VII.

§ 217. *To describe a square equal to the sum of two given squares,* AB^2 *and* AC^2.

CIII.

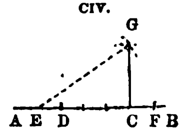


Make a right angle BAC, having its sides AB and AC equal to a side of each of the given squares. Join BC, and upon BC describe the square BHIC, which will be the square required.

As aBAC = 90°, $BC^2 = AB^2 + AC^2$ (§ 188).

Notes. 1. In the same manner, any number of squares may be successively added.

CIV.



§ 218. 2. The equality of 5^2 to $4^2 + 3^2$ furnishes an additional means of raising a perpendicular to a straight line AB at a given point C (§ 137). From C lay off, upon one side, 3 equal parts to D, and 4 to E; and, upon the other, 1 to F. Then, from E, with the radius EF, and from C, with the radius CD, describe arcs cutting each other in G. Join GC, which will be the perpendicular required. Since GE^2 is $= EC^2 + GC^2$, aECG = 90° (§ 203. b).

(f) Lat. complementum, from compleo, *to fill up.*

The unit CF may be made of any convenient length; or the equal parts may be taken from a scale.

Problem VIII.

§ 219. *To describe a square equal to the difference of two given squares*, BC^2 *and* AB^2.

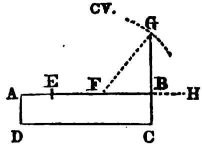
Draw AB (Fig. CIII.) equal to a side of the less square, and from A, draw AK ⊥ AB. From B as a centre, with a radius equal to a side of the greater square, describe an arc cutting AK in C. Upon AC describe the square ACDE, which will be the square required.

As aBAC $= 90°$, $AC^2 = BC^2 - AB^2$ (§ 190).

Problem IX.

§ 220. *To describe a square equal to a given rectangle* AB × BC.

CV.



From AB (the longer side) take AE = BC, and bisect EB in F. From F as a centre, with the radius FA, describe an arc cutting CB produced in G. Upon BG describe a square, which will be the square required.

AB is the sum of AF (= FG) and FB, and AE (= BC) is their difference. Therefore, $AB \times BC = FG^2 - FB^2$ (§ 185). But $FG^2 - FB^2 = BG^2$. $\therefore AB \times BC = BG^2$.

Notes. 1. The point F may also be obtained by producing AB to H, making BH = BC, and then bisecting AH.

§ 221. 2. As a square can be described equal to any rectangle, a rectangle equal to any triangle (§ 213), and a triangle equal to any other polygon (§ 212), it follows that a square can be described equal to any given polygon. This operation is called the *quadrature*[g] or *squaring* of the polygon.

(g) Lat. quadratūra, *squaring*.

3. As a triangle is equal to the rectangle of its base and half its altitude (§ 173), the side of a square equal to any given triangle may be immediately found by taking the base of the triangle in the place of AB (Fig. cv.), and half its altitude in the place of BH, and drawing BG $\perp$ AH to meet an arc described from F the middle of AH with the radius FA. BG will be the side required.

THE END.

DIAGRAMS.

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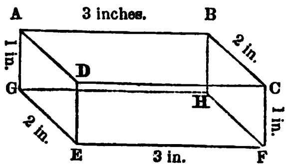

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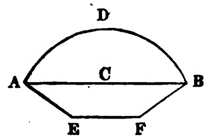

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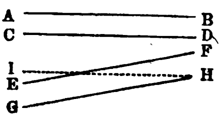

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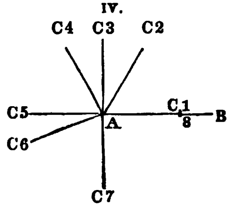

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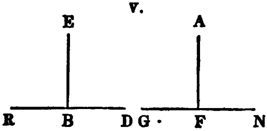

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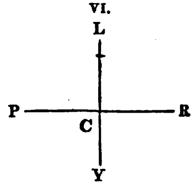

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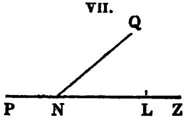

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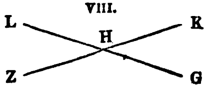

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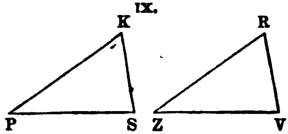

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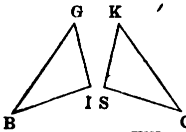

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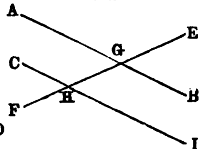

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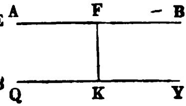

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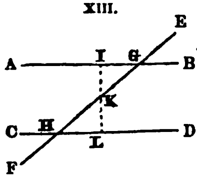

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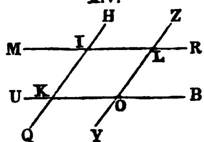

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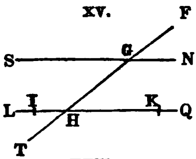

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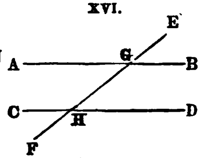

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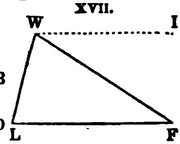

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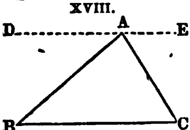

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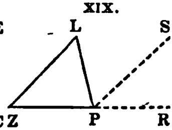

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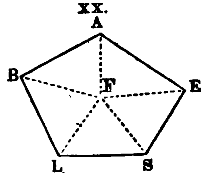

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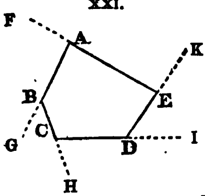

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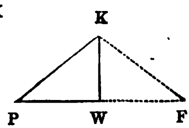

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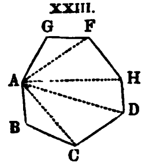

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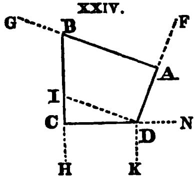

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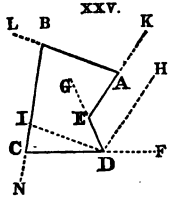

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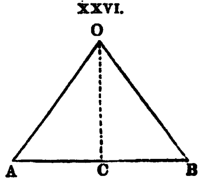

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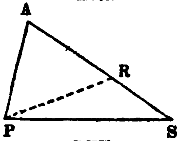

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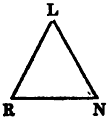

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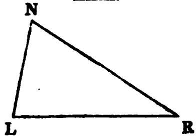

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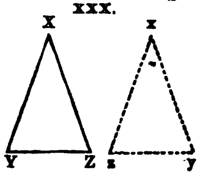

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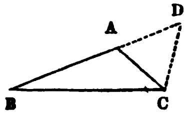

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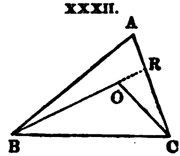

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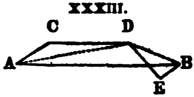

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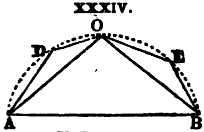

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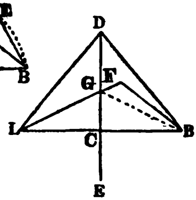

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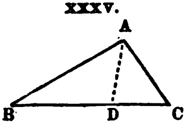

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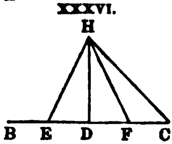

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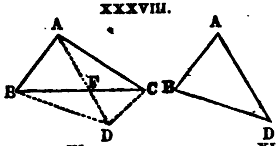

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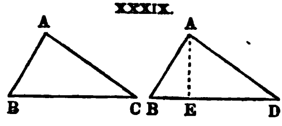

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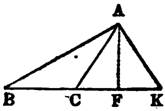

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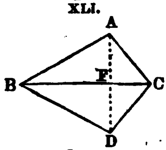

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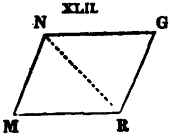

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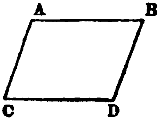

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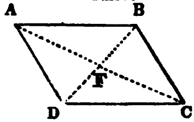

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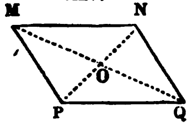

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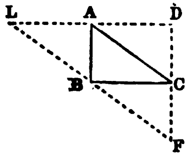

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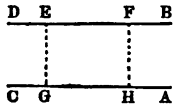

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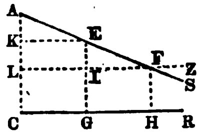

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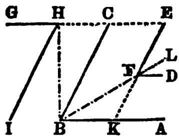

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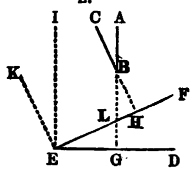

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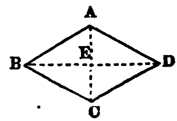

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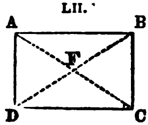

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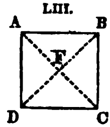

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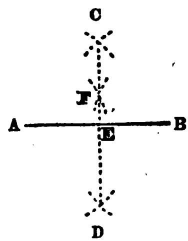

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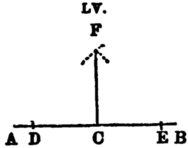

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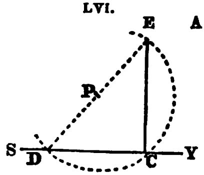

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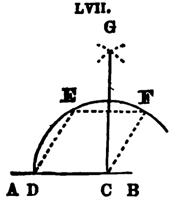

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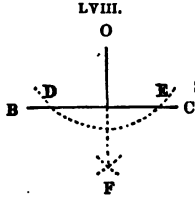

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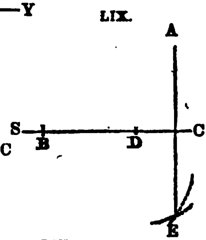

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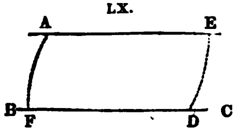

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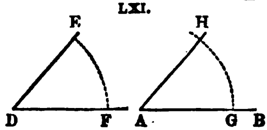

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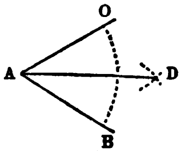

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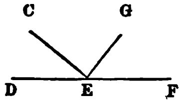

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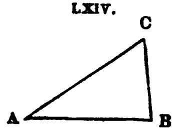

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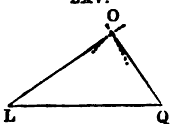

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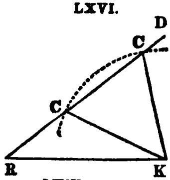

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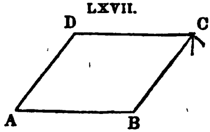

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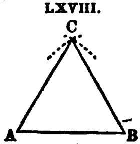

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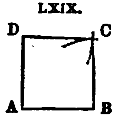

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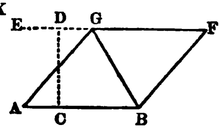

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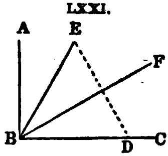

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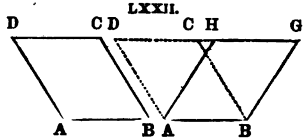

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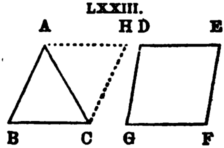

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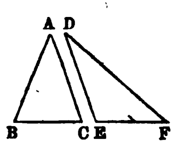

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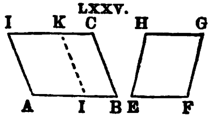

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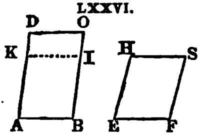

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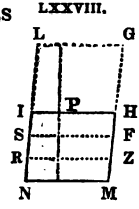

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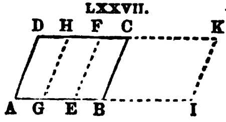

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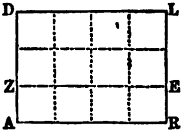

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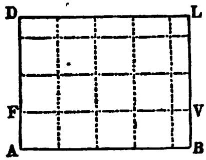

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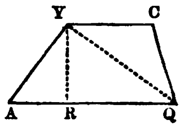

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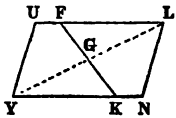

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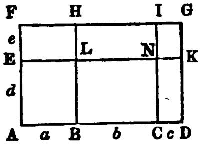

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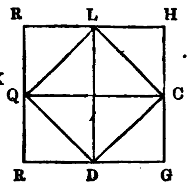

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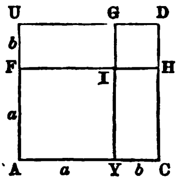

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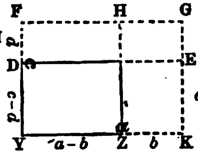

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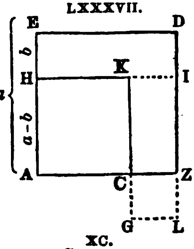

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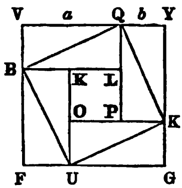

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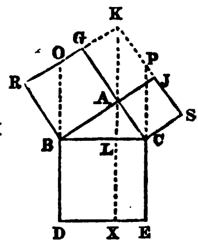

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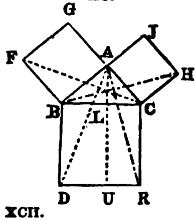

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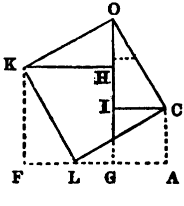

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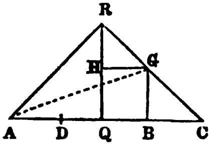

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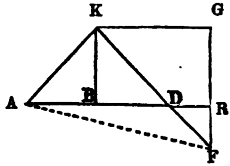

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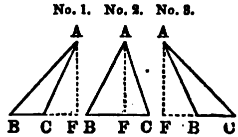

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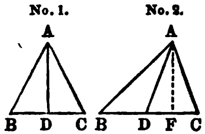

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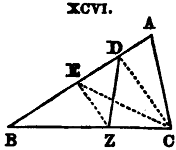

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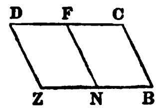

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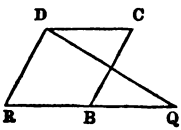

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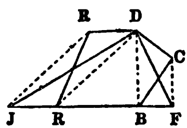

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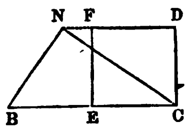

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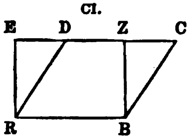

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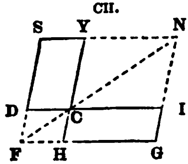

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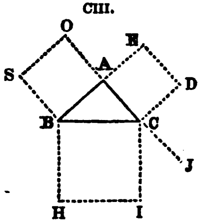

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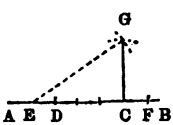

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